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December 11, 2012

Abstract

This paper studies the numerical approximation of periodic solutions for an exponentially stable linear hyperbolic equation in the presence of a periodic external force f. These approximations are obtained by combining a fixed point algorithm with the Galerkin method. It is known that the energy of the usual discrete models does not decay uniformly with respect to the mesh size. Our aim is to analyze this phenomenon's consequences on the convergence of the approximation method and its error estimates. We prove that, under appropriate regularity assumptions on f, the approximation method is always convergent. However, our error estimates show that the convergence's properties are improved if a numerically vanishing viscosity is added to the system. The same is true if the nonhomogeneous term f is monochromatic. To illustrate our theoretical results we present several numerical simulations with finite element approximations of the wave equation in one or two dimensional domains and with different forcing terms.

Mathematics Subject Classification (2010): 35B10, 65P99, 93C20

Key words: periodic solutions, dissipative hyperbolic system, Galerkin approximation, fixed point theorem, vanishing viscosity, error estimates.

1 Introduction

In order to present the purpose of our work, let us consider a general nonhomogeneous hyperbolic equation of the form

$$\ddot{u}(t) + Au(t) = f(t) \qquad (t > 0), \tag{1.1}$$

where A is a linear positive self-adjoint unbounded operator with compact resolvents in a Hilbert space H and $f : [0, \infty) \to H$ is a measurable function which can be viewed as an external force.

If the solutions of (1.1) are globally defined, it is interesting to study how does the force f affect their behavior as t goes to infinity. One of the most natural question is the following: Are the solutions $(u, \dot{u})$ of (1.1) bounded in the energy space $\mathcal{D}(A^{\frac{1}{2}}) \times H$ if f belongs to $\mathcal{C}_b([0, \infty); H)$, the set of continuous and bounded functions with values in H? If the answer is negative, we use to say that the *resonance phenomenon* occurs. In practice, the resonance can result in catastrophic failure of the object at resonance. The classic example of this is breaking a wine glass with sound at the precise resonant frequency of the glass. Avoiding resonance is a major concern in every building, tower and bridge construction project.

A particular case of bounded forces f are the periodic ones. If we suppose that the function f is periodic, it is well known that the boundedness of all trajectories corresponding to (1.1) and, consequently, the lack of resonance, is equivalent to the existence of a periodic solution of (1.1). Under the above hypothesis on the operator A, it is easy to see that unbounded solutions do exist. Indeed, it is sufficient to take $f(t) = 2\sqrt{\lambda}\, i\, e^{i\sqrt{\lambda}t}\phi$, where λ and ϕ are an eigenvalue and the corresponding eigenvector of A, respectively. Since $\lambda \in \mathbb{R}$, the function f is periodic. A particular solution of (1.1) is given by $u(t) = te^{i\sqrt{\lambda}t}\phi$, which is obviously unbounded. For more details, the interested reader is referred to [15].

Generally, a strong dissipative effect in the equation makes the resonance impossible and ensures the existence of periodic solutions. Therefore, we shall consider instead of (1.1) the following equation

$$\ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = f(t) \qquad (t > 0), \tag{1.2}$$

where A is like above and B is a bounded operator from another Hilbert space U into H ensuring *the uniform exponential decay of solutions*. More precisely, if we define the energy of a solution $(u, \dot{u})$ of (1.2) by

$$E(t) = \frac{1}{2}\|(u(t), \dot{u}(t))\|^2_{\mathcal{D}(A^{\frac{1}{2}})\times H}$$

and we suppose that $f = 0$, then there exist two positive constants M and ω such that the following inequality holds

$$E(t) \leqslant M^2 e^{-2\omega t} E(0), \tag{1.3}$$

for every solution $(u, \dot{u})$ of (1.2) with initial data $(u(0), \dot{u}(0)) = (u^0, u^1) \in \mathcal{D}(A^{\frac{1}{2}}) \times H$.

Under these hypotheses, for each $T-$periodic function $f \in \mathcal{C}_b([0, \infty); H)$, system (1.2) has exactly one $T-$periodic solution. To prove this, let us define the operator $\Lambda : \mathcal{D}(A^{\frac{1}{2}}) \times H \to \mathcal{D}(A^{\frac{1}{2}}) \times H$ which associates to each initial data $(u^0, u^1) \in \mathcal{D}(A^{\frac{1}{2}}) \times H$ the corresponding solution of (1.2) at time T,

$$\Lambda(u^0, u^1) = (u(T), \dot{u}(T)).$$

Now, we can use (1.3) to show that Λ has a unique fixed point $(\widehat{u}^0, \widehat{u}^1)$ which gives the periodic solution we were looking for. Indeed, (1.3) implies that Λ^n is a contraction for n sufficiently large and the existence of a unique fixed point for Λ is a consequence of Banach Fixed Point Theorem.

Note that this periodic solution is the global attractor of the system. Proving the existence of periodic solutions of (1.2) when f is a periodic function and giving a satisfactory numerical approximation of them represents two major tasks in the study of evolution of (1.2). We point out that the contractive properties of the operator Λ, which are very important for the convergence of the fixed point algorithm, depend on the exponent ω.

The aim of this paper is to present and analyze approximation methods for the periodic solutions for (1.2) and to offer precise error estimates in each case. We shall consider a space-discrete version of (1.2) obtained, for instance, from a finite element space discretization

$$\ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) = f_h(t), \tag{1.4}$$

where h denotes the mesh size and f_h is an approximation of the periodic force f. The operators A_h and B_h are discrete versions of the operators A and B, respectively, and will be introduced latter on. The solution $(u_h, \dot{u}_h)$ of (1.4) belongs to a finite dimensional space $V_h^2 = V_h \times V_h$. As in the continuous case, we can prove the existence of periodic solutions by showing that the operator $\Lambda_h : V_h^2 \to V_h^2$, which associates to each initial data (u_h^0, u_h^1) the corresponding solution of (1.4) at time T,

$$\Lambda_h(u_h^0, u_h^1) = (u_h(T), \dot{u}_h(T)),$$

has a fixed point $(\widehat{u}_h^0, \widehat{u}_h^1)$. In this paper, we shall analyze the following algorithm for the approximation of the periodic solutions of (1.2):

- Choose $h > 0$ small and $(u_h^0, u_h^1) \in V_h^2$;
- Choose an integer $N > 0$ and a precision $\epsilon > 0$;
- Compute $\Lambda_h^n(u_h^0, u_h^1)$ until

$$\|\Lambda_h^n(u_h^0, u_h^1) - \Lambda_h^{n-1}(u_h^0, u_h^1)\| < \epsilon$$
$$\text{or}$$
$$n > N;$$

- Approximate $(\widehat{u}^0, \widehat{u}^1)$ by $\Lambda_h^n(u_h^0, u_h^1)$.

Our error estimates will show how the numbers N and ϵ should be chosen, depending on the error of the Galerkin scheme used for the space discretization. Note that the above algorithm combines two approximation processes: one for computing the solution of the wave equation (1.2) at time T, given by $\Lambda_h(u_h^0, u_h^1)$, and another one for determining the fixed point of Λ_h. While the first numerical process depends on the discrete operators A_h and B_h, the second one is governed by the contractive properties of Λ_h. Most often, improving one of these approximations leads to the deterioration of the other one. Let us briefly explain this phenomenon. When considering numerical discretization schemes for wave equations, it is well known that most of them do not preserve the uniform (with respect to the mesh-size h) decay property of the solutions of the continuous wave equation (1.2). Indeed, as remarked in [24, 25] (see, also, [4, 17, 27, 28]), due to the existence of high frequency spurious solutions whose (group) velocity of propagation is of the order of h, the energy of the discrete solution $(u_h, \dot{u}_h)$ does not have a uniform exponential decay. This means that the discrete energy of solutions defined by

$$E_h(t) = \frac{1}{2}\|(u_h(t), \dot{u}_h(t))\|_{V_h^2}^2$$

verifies the inequality

$$E_h(t) \leqslant M_h^2 e^{-2\omega(h)t} E_h(0), \tag{1.5}$$

with an exponential decay rate $\omega(h)$ which may tend to zero as h goes to zero. A consequence of this phenomenon is a degeneracy of the contractive properties of the operator Λ_h as h tends to zero. Since the contractive properties of Λ_h are fundamental in the convergence of the fixed point method presented above, the entire approximation method used for the periodic solutions of (1.2) will be affected by this phenomenon.

Three main problems will be investigated in this paper:

(P1) Does the family $\left(\Lambda_h^n(u_h^0, u_h^1)\right)_{h>0,\, n\geqslant 0}$ converge to the fixed point $(\widehat{u}^0, \widehat{u}^1)$ of Λ when h tends to zero and n tends to infinity and, if it does, at which rate?

(P2) Does the family $\left(\widehat{u}_h^0, \widehat{u}_h^1)\right)_{h>0}$ of fixed points of the discrete operators Λ_h converges to the fixed point $(\widehat{u}^0, \widehat{u}^1)$ of Λ when h goes to zero and, if it does, at which rate?

(P3) How do the two convergence properties mentioned above change if a numerical vanishing viscosity is introduced in the equation or if special monochromatic nonhomogeneous terms f are considered?

In Section 6 we shall give a positive answer to problem (P1). Indeed, Theorem 6.4 proves that, under the additional regularity condition $f \in W^{1,1}(0,T;\mathcal{D}(A^{\frac{1}{2}}))$, the family $\left(\Lambda_h^n(u_h^0, u_h^1)\right)_{h>0,\, n\geqslant 0}$ converges to $(\widehat{u}^0, \widehat{u}^1)$ with a velocity of order $h^\theta \ln\left(\frac{1}{h}\right)$, where h^θ is the error of the Galerkin method considered in (1.4). However, due precisely to the nonuniform decay of the discrete energy in this case, we cannot guarantee the convergence of $\left(\widehat{u}_h^0, \widehat{u}_h^1\right)_{h>0}$ to $(\widehat{u}^0, \widehat{u}^1)$ and a positive answer to (P2).

As we have mentioned before, the lack of uniform exponential decay in (1.5) is due to the high spurious numerical frequencies and their filtering may be useful. A convenient and natural method of filtering consists in adding a vanishing numerical viscosity depending on a small parameter $\varepsilon = h^\eta$ which tends to zero as h goes to zero. The parameter $\eta > 0$ allows us to control the amount of extra-dissipation introduced in the system. Hence, instead of (1.4) we can consider the following discretization

$$\ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) + h^\eta A_h \dot{u}_h(t) = f_h(t) \qquad (t > 0). \tag{1.6}$$

It is known that this procedure ensures the uniform exponential decay of (1.6) (see, for instance, [12, 13, 19, 21, 24, 25]). More precisely, the solution $(u_h, \dot{u}_h)$ of (1.6) verifies (1.5) with constants M and ω independent of h. Since the term $h^\eta A_h \dot{u}_h(t)$ enforces the dissipation, the exponential decay of the solutions of (1.6) gets better when η decreases. However, in order to preserve the precision of the numerical scheme for (1.2), the parameter η cannot be too small. Our error analysis allows us to conclude in Section 5 that the optimal value of ε is h^θ, where θ is the precision of the finite element discretization. With this particular choice, we use a result of uniform decay obtained in [12] to show that the contractive properties of the operator Λ_h are uniform with respect to h. Consequently, positive answers to both problems (P1) and (P2) are given in Theorem 6.8 and Corollary 6.9. Moreover, we show that the velocity of convergence is of order h^θ, the same as the Galerkin scheme. Hence, the vanishing viscosity not only ensures the convergence of the discrete fixed points of Λ_h to the continuous one of Λ but, at the same time, improves the convergence velocity of the iterative family $\left(\Lambda_h^n(u_h^0, u_h^1)\right)_{h>0,\, n\geqslant 0}$.

Our paper is related to [5, 8, 9, 14, 29], where the approximation of the outgoing solutions of Helmholtz's equation in an exterior domain $\Omega \subset \mathbb{R}^n$ is studied. For numerical reasons, the unbounded domain Ω has to be limited by introducing an artificial boundary Γ with a Sommerfeld condition. This problem is reduced to find the periodic solution of a wave equation with boundary dissipation and a source term f which is *monochromatic*, i. e. of the form $f(t) = e^{i\varsigma t}\phi$, where $\varsigma \in \mathbb{R}$ and ϕ is a given function. The method used in these papers consists in obtaining the periodic solutions as minimizers of some suitable functionals. Although different from our approach, the approximation process in [8, 9, 14] is also affected by the fact that the exponential decay of the energy corresponding to the discrete system (1.4) is, in general, not uniform with respect to the mesh size. Indeed, this produces at the same time the lack of coerciveness of the functional to be minimized in [8, 9, 14] and the lack of uniform contraction property of our operator Λ_h. Let us also mention that, although characterized by a higher memory consumption, a valid alternative to the vanishing viscosity method analyzed here is the minimization of the modified functional introduced and analyzed in [5] and whose discrete version is studied in [29]. In Section 7 we consider the above mentioned special case of monochromatic source term f. We show that both problems (P1) and (P2) have positive answers, even if no viscosity is added and (1.4) is used. This result is a consequence of the lack of high frequencies oscillations of the monochromatic term f and it is in consonance with the good numerical approximations obtained in [8, 9, 14].

The remaining part of the paper is organized as follows. In Section 2 we present some basic properties of system (1.2) and prove the existence of its periodic solutions. The first discrete version (1.4) of (1.2) is considered in Section 3 and the existence of discrete periodic solutions is proved. In Section 4 we analyze the discrete problem (1.6) resulted by adding the numerical viscosity. Sections 5 and 6 are devoted to prove the main error estimates in the approximation of periodic solutions with or without viscosity. In Section 7 monochromatic periodic forces f are considered and analyzed. Finally, in Section 8 we present some numerical simulations which are based on and confirm the theoretical results.

2 Existence of periodic solutions

In order to give the precise statement of our results we need some notation. Let H be a Hilbert space and assume that $A : \mathcal{D}(A) \to H$ is a self-adjoint, strictly positive operator with compact resolvent. Then, according to classical results, the operator A is diagonalizable with an orthonormal basis $(\varphi_k)_{k\geqslant 1}$ of eigenvectors and the corresponding family of positive eigenvalues $(\lambda_k)_{k\geqslant 1}$ satisfies $\lim_{k\to\infty} \lambda_k = \infty$. Moreover, we have

$$\mathcal{D}(A) = \left\{ z \in H \;\middle|\; \sum_{k\geqslant 1} \lambda_k^2 \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\},$$

and

$$Az = \sum_{k\geqslant 1} \lambda_k \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A)).$$

For $\alpha \geqslant 0$ the operator A^α is defined by

$$\mathcal{D}(A^\alpha) = \left\{ z \in H \;\middle|\; \sum_{k\geqslant 1} \lambda_k^{2\alpha} \left|\langle z, \varphi_k\rangle\right|^2 < \infty \right\}, \tag{2.7}$$

and

$$A^\alpha z = \sum_{k\geqslant 1} \lambda_k^\alpha \langle z, \varphi_k\rangle \varphi_k \qquad (z \in \mathcal{D}(A^\alpha)).$$

For every $\alpha \geqslant 0$ we denote by H_α the space $\mathcal{D}(A^\alpha)$ endowed with the inner product

$$\langle \varphi, \psi \rangle_\alpha = \langle A^\alpha \varphi, A^\alpha \psi \rangle \qquad (\varphi,\ \psi \in H_\alpha).$$

The induced norm is denoted by $\|\cdot\|_\alpha$, except in the case $\alpha = 0$ when, for simplicity, $\|\cdot\|$ will be used. From the above facts it follows that for every $\alpha \geqslant 0$ the operator A is a unitary operator from $H_{\alpha+1}$ onto H_α and A is strictly positive on H_α.

Let U be another Hilbert space and let $B \in \mathcal{L}(U, H)$ be an input operator. Consider the system

$$\ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = 0 \qquad (t \geqslant 0) \tag{2.8}$$

$$u(0) = u_0 \qquad \dot{u}(0) = u_1. \tag{2.9}$$

For every $(u_0, u_1) \in H_{\frac{1}{2}} \times H$, the energy corresponding to (2.8)-(2.9) is defined by

$$E(t) = \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \|\dot{u}(t)\|^2\right) \tag{2.10}$$

and verifies

$$E(0) - E(t) = \int_0^t \|B^*\dot{u}(s)\|_U^2 ds. \tag{2.11}$$

We remark that (2.11) shows that the energy of each solution of (2.8)-(2.9) is non-increasing. In this paper we assume furthermore that system (2.8)-(2.9) is *exponentially (or uniformly) stable*, i. e., there exist two positive constants M and ω such that

$$E(t) \leqslant M^2 E(0) e^{-2\omega t} \qquad (t \geqslant 0, \qquad (u_0, u_1) \in H_{\frac{1}{2}} \times H). \tag{2.12}$$

Let us now consider $f \in \mathcal{C}([0, \infty); H)$ and the non-homogeneous system

$$\begin{cases} \ddot{u}(t) + Au(t) + BB^*\dot{u}(t) = f(t) \qquad (t > 0), \\ u(0) = u_0, \ \dot{u}(0) = u_1. \end{cases} \tag{2.13}$$

The aim of this section is to show the existence of a periodic solution of (2.13), under the assumption that f is a time periodic function of period T

$$f(t + T, \cdot) = f(t, \cdot) \qquad (t \geqslant 0) \tag{2.14}$$

and to study its main properties. The results presented in this section are not new (see, for instance, Haraux [15] for an excellent introduction to the topic of periodic solutions for nonlinear hyperbolic problems) but we have chosen to include them here in order to fix the notation and to make the paper easier to read.

Firstly, let us deal with the general initial value problem (2.13). In order to do that, it is convenient to write (2.13) as a differential equation of first order in time. Introducing the new variable $v = \dot{u}$, we obtain that (2.13) may be equivalently written as

$$\dot{U}(t) = \mathbb{A}U(t) + F(t) \qquad (t > 0) \tag{2.15}$$

$$U(0) = U_0 \tag{2.16}$$

where

$$U(t) = \begin{bmatrix} u(t) \\ v(t) \end{bmatrix}, \ \mathbb{A} = \begin{bmatrix} 0 & I \\ -A & -BB^* \end{bmatrix}, \ F(t) = \begin{bmatrix} 0 \\ f(t) \end{bmatrix} \ \text{ and } \ U^0 = \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

For each $\alpha \geqslant 0$, we consider the Hilbert space $X_\alpha = H_{\alpha+\frac{1}{2}} \times H_\alpha$ with its canonical inner product

$$\left\langle \begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}, \begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix} \right\rangle_{X_\alpha} = \langle A^{\frac{1}{2}}\varphi_1, A^{\frac{1}{2}}\varphi_2\rangle_\alpha + \langle \psi_1, \psi_2\rangle_\alpha \qquad \left(\begin{bmatrix}\varphi_1\\ \psi_1\end{bmatrix}, \begin{bmatrix}\varphi_2\\ \psi_2\end{bmatrix} \in X_\alpha\right).$$

Here and henceforth X_0 will be simply denoted by X. In the case $F = 0$, it is known that (2.15)-(2.16) defines a well-posed dynamical system in the space X. More precisely, for each $U^0 \in X$, the solution U of (2.15)-(2.16) is given by

$$U(t) = \mathbb{S}(t)U^0 \qquad (t \geqslant 0)\,, \tag{2.17}$$

where $\mathbb{S}$ is the contraction semigroup on X generated by the unbounded operator $\mathbb{A}$ in X with domain $\mathcal{D}(\mathbb{A}) = X_{\frac{1}{2}}$. Assumption (2.12) ensures the exponential stability of the semigroup $\mathbb{S}$ in X, i. e., there exists two positive constants M and ω such that

$$\|\mathbb{S}(t)\|_{\mathcal{L}(X)} \leqslant Me^{-\omega t} \qquad (t \geqslant 0). \tag{2.18}$$

The same decay rate of the semigroup is achieved in $X_{\frac{1}{2}}$ and, under additional hypothesis on the regularity of the operator BB^*, in X_1. Here and in the sequel the semigroup $\mathbb{S}$ in X_α means the semigroup generated by the restriction (or the extension) of the unbounded operator $\mathbb{A}$ to X_α with the domain $X_{\alpha+\frac{1}{2}}$. We have the following result.

Proposition 2.1. *With the above notation, assume that* (2.12) *is verified. Then the semigroup* $\mathbb{S}$ *is exponentially stable in* X *and* $X_{\frac{1}{2}}$*. If, in addition,* $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$*, then* $\mathbb{S}$ *is an exponentially stable semigroup in* X_1*, too.*

Proof. The fact that $\mathbb{S}$ is an exponentially stable semigroup in X follows directly from (2.12). To show the same property in $X_{\frac{1}{2}}$, note that the graph norm of $\mathbb{A}$ is equivalent to the standard norm of $X_{\frac{1}{2}}$. Therefore, for each $U^0 \in X_{\frac{1}{2}}$, we have that

$$\|\mathbb{S}(t)U^0\|_{X_{\frac{1}{2}}} \leqslant C\left(\|\mathbb{A}\mathbb{S}(t)U^0\|_X + \|\mathbb{S}(t)U^0\|_X\right) = C\left(\|\mathbb{S}(t)\mathbb{A}U^0\|_X + \|\mathbb{S}(t)U^0\|_X\right)$$

for some positive constant and the exponential stability of $\mathbb{S}$ in $X_{\frac{1}{2}}$ follows from (2.12).

To prove the exponential stability of $\mathbb{S}$ in X_1, note that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ implies that $\mathcal{D}(\mathbb{A}^2) = X_1$ and that the graph norm of $\mathbb{A}^2$ is equivalent to the standard norm in X_1. Now the assertion can be easily obtained by repeating the above argument in X_1. □

We pass now to the study of the non-homogeneous problem. The following result is well-known and may be found, for instance, in [10].

Proposition 2.2. *Let* $\alpha \in \{0, \frac{1}{2}, 1\}$ *and* $f \in \mathcal{C}([0,\infty); H_\alpha)$*. Then, for any* $U^0 \in X_\alpha$*, problem* (2.15)-(2.16) *has a unique mild solution* $U \in \mathcal{C}([0,\infty); X_\alpha)$ *given by*

$$U(t) = \mathbb{S}(t)U^0 + \int_0^t \mathbb{S}(t-s)F(s)\,ds \qquad (t \geqslant 0). \tag{2.19}$$

Moreover, if $f \in W^{1,1}_{loc}([0,\infty); H_\alpha)$ *or* $f \in L^1_{loc}\left([0,\infty); H_{\alpha+\frac{1}{2}}\right)$ *and* $U^0 \in X_{\alpha+\frac{1}{2}}$*, then system* (2.15)-(2.16) *has a unique classical solution* $U \in \mathcal{C}\left([0,\infty); X_{\alpha+\frac{1}{2}}\right) \cap \mathcal{C}^1([0,\infty); X_\alpha)$ *verifying*

$$\|U(t)\|_{X_{\alpha+\frac{1}{2}}} + \|\dot{U}(t)\|_{X_\alpha} \leqslant C\left[\|U^0\|_{X_{\alpha+\frac{1}{2}}} + \|f\|_{W^{1,1}(0,T;H_\alpha)}\right] \qquad (0 \leqslant t \leqslant T), \tag{2.20}$$

where C *is a positive constant and* $T > 0$ *is arbitrary.*

We have now all the ingredients needed to show the existence of periodic solutions for (2.13).

Theorem 2.3. *Assume that $f \in \mathcal{C}([0,\infty);H)$ satisfies (2.14). Then there exists a unique $\widehat{U}^0 \in X$ such that the corresponding mild solution $\widehat{U}$ of* (2.15) *in X with initial data $\widehat{U}^0$ is $T-$periodic.*

If, in addition, $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $f \in W^{1,1}_{loc}([0,\infty); H_{\frac{1}{2}})$ or $f \in L^1_{loc}([0,\infty);H_1)$, then there exists a unique $\widehat{U}^0 \in X_1$ such that the corresponding classical solution $\widehat{U}$ of (2.15) *in X_1 with initial data $\widehat{U}^0$ is $T-$periodic.*

Proof. The proof is based on a contraction argument. We introduce the map $\Lambda : X \to X$ defined by

$$\Lambda U^0 = \mathbb{S}(T)U^0 + \int_0^T \mathbb{S}(T-s)F(s)\,ds. \tag{2.21}$$

From Proposition 2.1 we obtain that, for any $U^0 \in X$,

$$\|\mathbb{S}(t)U^0\|_X \leqslant M\,e^{-\omega t}\|U^0\|_X \qquad (t \geqslant 0). \tag{2.22}$$

Moreover, taking into account the semigroup properties and the periodicity of F, it is easy to see by induction that the following identity holds for any $n \in \mathbb{N}$

$$\Lambda^n U^0 = S(nT)U^0 + \int_0^{nT} \mathbb{S}(nT-s)F(s)\,ds. \tag{2.23}$$

Using (2.22) and (2.23) we have that, for any U^0 and U^1 in X,

$$\|\Lambda^n U^0 - \Lambda^n U^1\|_X = \|S(nT)(U^0 - U^1)\|_X \leqslant Me^{-n\omega T}\|U^0 - U^1\|_X.$$

It follows that, for n sufficiently large, Λ^n is a contraction on X and therefore there exists a unique $\widehat{U}^0 \in X$ such that $\Lambda^n \widehat{U}^0 = \widehat{U}^0$. Consequently, $\Lambda^n(\Lambda\widehat{U}^0) = \Lambda(\Lambda^n\widehat{U}^0) = \Lambda\widehat{U}^0$ and, by the uniqueness of the fixed point of Λ^n, we obtain $\Lambda\widehat{U}^0 = \widehat{U}^0$. This means that the mild solution $\widehat{U}$ of (2.15) with the initial data $\widehat{U}^0$ satisfies $\widehat{U}(T) = \widehat{U}(0)$. Using the semigroup property we see easily that $\widehat{U}(t+T) = \widehat{U}(t)$, for all $t \geqslant 0$ and $\widehat{U}$ is a $T-$periodic solution in X of (2.15).

Now, suppose that $f \in W^{1,1}_{loc}([0,\infty); H_{\frac{1}{2}})$ of $f \in L^1_{loc}([0,\infty);H_1)$. In this case we can repeat the previous arguments considering $\Lambda : X_1 \to X_1$, defined by (2.21). According to Proposition 2.2, this map is well-defined. Moreover, from Proposition 2.1 we obtain that, for any $U^0 \in X_1$, the following inequality holds

$$\|\mathbb{S}(T)U^0\|_{X_1} \leqslant M\,e^{-\omega T}\|U^0\|_{X_1} \qquad (t \geqslant 0).$$

Arguing as in the first part of the proof, we conclude that there exists a unique $\widehat{U}^0 \in X_1$ such that the classical solution $\widehat{U} \in \mathcal{C}([0,\infty);X_1) \cap \mathcal{C}^1([0,\infty);X_{\frac{1}{2}})$ of (2.15) with initial data $\widehat{U}^0$ is T-periodic. $\square$

The periodic solution found in Theorem 2.3 has an important property which makes it special: it is the global attractor of the dynamical system (2.15). More precisely, we have the following property.

Proposition 2.4. *Suppose that $f \in \mathcal{C}[0,\infty; H)$ and verifies (2.14). If $U \in \mathcal{C}([0,\infty); X)$ is a T-periodic solution of* (2.15) *and $V \in \mathcal{C}([0,\infty); X)$ is any other solution of* (2.15) *with initial condition $V(0) = V_0$ then*

$$\lim_{t\to\infty} \|U(t) - V(t)\|_X = 0. \tag{2.24}$$

As a consequence, system (2.15) *has at most one $T-$periodic solution in $\mathcal{C}([0,\infty); X)$.*

Proof. Let $W = U - V$. It follows that $W \in \mathcal{C}([0,\infty); X)$ and verifies

$$W(t) = \mathbb{S}(t)(U(0) - V(0)), \qquad \forall t \geqslant 0.$$

Now, (2.24) follows from (2.18).

If V is another $T-$periodic solution we deduce that $W = U - V$ is $T-$periodic too. From (2.24) we obtain that $U = V$ and the proof ends. $\square$

In the remaining part of this article we study the numerical approximation of the periodic solution $\widehat{U}$ given by Theorem 2.3.

3 Discrete problem: first method

Assume that there exists a family $(V_h)_{h>0}$ of finite dimensional subspaces of $H_{\frac{1}{2}}$ and that there exist $\theta > 0$, $h^* > 0$, $C_0 > 0$ such that, for every $h \in (0, h^*)$,

$$\|\pi_h\varphi - \varphi\|_{\frac{1}{2}} \leq C_0\, h^\theta \|\varphi\|_1 \qquad (\varphi \in H_1), \tag{3.1}$$

$$\|\pi_h\varphi - \varphi\| \leq C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}), \tag{3.2}$$

where π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h. Assumptions (3.1)-(3.2) are, in particular, satisfied when finite elements are used for the approximation of Sobolev spaces. The inner product in V_h is the restriction of the inner product on H and it is still denoted by $\langle\cdot,\cdot\rangle$. $N(h)$ denotes the dimension of V_h. We define the linear operator $A_h \in \mathcal{L}(V_h)$ by

$$\langle A_h\varphi_h, \psi_h\rangle = \langle A^{\frac{1}{2}}\varphi_h, A^{\frac{1}{2}}\psi_h\rangle \qquad (\varphi_h,\ \psi_h \in V_h). \tag{3.3}$$

The operator A_h is clearly symmetric and strictly positive.

Denote $U_h = B^*V_h \subset U$ and define the operators $B_h \in \mathcal{L}(U, H)$ by

$$B_h u = \widetilde{\pi}_h B u \qquad (u \in U), \tag{3.4}$$

where $\widetilde{\pi}_h$ is the orthogonal projection from H onto V_h. Note that $\operatorname{Ran} B_h \subset V_h$. As well-known, since it is an orthogonal projector, the operator $\widetilde{\pi}_h \in \mathcal{L}(H)$ is self-adjoint. Moreover, from (3.2) we deduce that

$$\|\varphi - \widetilde{\pi}_h\varphi\| \leqslant \|\varphi - \pi_h\varphi\| \leqslant C_0\, h^\theta \|\varphi\|_{\frac{1}{2}} \qquad (\varphi \in H_{\frac{1}{2}}). \tag{3.5}$$

The adjoint $B_h^* \in \mathcal{L}(H, U)$ of B_h is

$$B_h^*\varphi = B^*\widetilde{\pi}_h\varphi \qquad (\varphi \in H). \tag{3.6}$$

Since $U_h = B^* V_h$, from (3.6) it follows that $\operatorname{Ran} B_h^* = U_h$ and that

$$\langle B_h^* \varphi_h, B_h^* \psi_h \rangle_U = \langle B^* \varphi_h, B^* \psi_h \rangle_U \qquad (\varphi_h,\ \psi_h \in V_h). \tag{3.7}$$

The above assumptions imply that, for every $h^* > 0$, the family $\left(\|B_h\|_{\mathcal{L}(U_h, H)} \right)_{h \in (0, h^*)}$ is bounded.

Now, we consider the following semi-discrete scheme for (2.13)

$$\begin{cases} \ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) = f_h(t), & (t > 0) \\ u_h(0) = u_{0h}, \ \dot{u}_h(0) = u_{1h}, \end{cases} \tag{3.8}$$

where $f_h \in \mathcal{C}([0, \infty); V_h)$ and $(u_{0h}, u_{1h}) \in V_h^2$.

In (3.8) f_h and (u_{0h}, u_{1h}) stand for approximations of f and (u_0, u_1) in (2.8), respectively. A more precise choice of these approximations will be given latter on.

We consider the product space $X_h = V_h \times V_h$ equipped with the inner product

$$\left\langle \begin{bmatrix} \varphi_{1h} \\ \psi_{1h} \end{bmatrix}, \begin{bmatrix} \varphi_{2h} \\ \psi_{2h} \end{bmatrix} \right\rangle_{X_h} = \langle A_h^{\frac{1}{2}} \varphi_{1h}, A_h^{\frac{1}{2}} \varphi_{2h} \rangle + \langle \psi_{1h}, \psi_{2h} \rangle,$$

and we denote by $\| \cdot \|_{X_h}$ the corresponding norm in X_h. Using the new variable $v_h = \dot{u}_h$, system (3.8) can be rewritten in the Hilbert space X_h as

$$\dot{U}_h(t) = \mathbb{A}_h U_h(t) + F_h(t), \qquad (t > 0) \tag{3.9}$$

$$U_h(0) = U_h^0, \tag{3.10}$$

where $\mathbb{A}_h = \mathbb{A}_h^1 - \mathbb{B}_h$ and

$$U_h(t) = \begin{bmatrix} u_h(t) \\ v_h(t) \end{bmatrix}, \ \mathbb{A}_h^1 = \begin{bmatrix} 0 & I \\ -A_h & 0 \end{bmatrix}, \ \mathbb{B}_h = \begin{bmatrix} 0 & 0 \\ 0 & B_h B_h^* \end{bmatrix}, \ F_h(t) = \begin{bmatrix} 0 \\ f_h(t) \end{bmatrix}, \ U_h^0 = \begin{bmatrix} u_{0h} \\ u_{1h} \end{bmatrix}.$$

Note that $\mathbb{A}_h^1$ is a skew-adjoint operator on X_h and $\mathbb{A}_h \in \mathcal{L}(X_h)$ is the infinitesimal generator of a uniformly continuous semigroup of linear bounded operators $\mathbb{S}_h(t) = e^{t\mathbb{A}_h}$ on X_h. Moreover, since $\mathbb{A}_h$ is m-dissipative, the semigroup $\mathbb{S}_h$ is of contractions in X_h. The following result is a standard one in ordinary differential equations.

Proposition 3.1. *Let $h > 0$ and $f_h \in \mathcal{C}([0, \infty); V_h)$. Then, for any $U_h^0 \in X_h$, system* (3.9)-(3.10) *has a unique solution $U_h \in \mathcal{C}^1([0, \infty); X_h)$ given by*

$$U_h(t) = \mathbb{S}_h(t) U_h^0 + \int_0^t \mathbb{S}_h(t - s) F_h(s)\, ds \qquad (t \geqslant 0). \tag{3.11}$$

Let $(\varphi_{hn})_{1 \leqslant n \leqslant N(h)}$ and $(\lambda_{hn})_{1 \leqslant n \leqslant N(h)}$ be the set of eigenvectors normalized in V_h and the corresponding eigenvalues of the operator $A_h^{\frac{1}{2}}$, respectively. A normalized basis of the product space V_h^2 is given by the set of eigenvectors of the skew-adjoint operator $\mathbb{A}_h^1$,

$$\Phi_{hn} = \frac{1}{\sqrt{2}} \begin{bmatrix} \varphi_{h|n|} \\ i \operatorname{sgn}(n) \lambda_{h|n|} \varphi_{h|n|} \end{bmatrix} \qquad (1 \leqslant |n| \leqslant N(h)). \tag{3.12}$$

The discrete energy corresponding to (3.8) is defined by

$$E_h(t) = \frac{1}{2}\left(\|A_h^{\frac{1}{2}}u_h\|^2 + \|\dot{u}_h\|^2\right). \tag{3.13}$$

If $f_h = 0$, then taking the inner product by $\dot{u}_h$ in equation (3.8), we deduce that

$$\frac{dE_h}{dt}(t) = -\,\|B^*\dot{u}_h(t)\|_U^2 \qquad (t \geqslant 0).$$

Thus, if $f_h = 0$, the energy E_h of (3.8) is non increasing. In the sequel we shall suppose that the following hypothesis holds

$$\lim_{t\to\infty} E_h(t) = 0. \tag{3.14}$$

Remark 3.2. *Since* (3.8) *is a finite dimensional system, it is easy to show that* (3.14) *is equivalent to the following property*

$$B_h^*\varphi_{hn} \neq 0 \qquad (1 \leqslant n \leqslant N). \tag{3.15}$$

There are cases in which (3.15) *does not hold in spite of the fact that the continuous counterpart, $B^*\varphi_n \neq 0$, is verified for each $n \geqslant 1$.*

An illustrative example is provided by the restriction to ω operator $B = \chi_\omega$ defined in $L^2(\Omega)$ and minus the Dirichlet Laplace operator in $L^2(\Omega)$, $A = -\Delta$. Ω is a nonempty open subset of $\mathbb{R}^d$ and ω a nonempty open subset of Ω. The corresponding discrete operators B_h and A_h are the restriction to the grid points from ω and minus the finite-difference Laplacian Δ_h, respectively. If Ω is the unit square, $\Omega = (0,1)\times(0,1)$, and ω a neighborhood of a part of the boundary of the form (Figure 1)

$$\omega = \left\{(x,y) \mid 0 < x < \varrho,\ 0 < y < \frac{1}{2} + \varrho\right\} \cup \left\{(x,y) \mid 1-\varrho < x < 1,\ \frac{1}{2} - \varrho < y < 1\right\} \cup$$
$$\left\{(x,y) \mid 0 < y < \varrho,\ 0 < x < \frac{1}{2} + \varrho\right\} \cup \left\{(x,y) \mid 1-\varrho < y < 1,\ \frac{1}{2} - \varrho < x < 1\right\},$$

where ϱ is any positive small number, then there exists eigenvectors φ_{hn} of Δ_h with the property that $B_h\varphi_{hn} = 0$ (see [28, §9.2, Problem 3]).

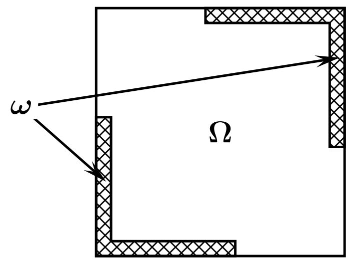


Figure 1: Domains Ω and ω described in Remark 3.2

On the contrary, in the simpler case $\Omega = (0,1)$, it is easy to see that (3.15) *does hold, indistinctly of the choice of ω. As mentioned in [28, §9.2, Problem 3], getting optimal geometric conditions on the set ω to ensure* (3.15) *in the multidimensional framework is an interesting and widely open subject of research.*

Finally, let us mention that a possibility to deal with the case in which (3.15) *is not verified consists in filtering the high frequencies which violate* (3.15)*. More details in this direction will be given at the end of Section 7.*

We have the following result concerning the decay of the discrete energy (3.13).

Proposition 3.3. *Let $h > 0$. Suppose that $f_h \equiv 0$ and that* (3.14) *holds. There exist two positive constants M independent of h and $\omega = \omega(h)$ (depending on h) such that*

$$E_h(t) \leqslant M^2 E_h(0) e^{-2\omega(h)t} \qquad (t \geqslant 0 \qquad (u_{0h}, u_{1h}) \in V_h^2). \tag{3.16}$$

Proof. Given $\tau > 0$, there exists a constant $K_{\tau,h} > 0$ such that the following inequality holds

$$\int_0^\tau \|B_h^* \dot{w}_h(t)\|_U^2 \,\mathrm{d}t \geqslant K_{\tau,h} \left(\|A_h^{\frac{1}{2}} w_{0h}\|^2 + \|w_{1h}\|^2 \right), \qquad (w_{0h}, w_{1h} \in V_h) \tag{3.17}$$

where $(w_h, \dot{w}_h)$ is the solution of

$$\begin{cases} \ddot{w}_h(t) + A_h w_h(t) = 0, & (t \in (0,\tau)) \\ w_h(0) = w_{0h}, \ \dot{w}_h(0) = w_{1h}. \end{cases} \tag{3.18}$$

Inequality (3.17) follows from the fact that the map $(w_{0h}, w_{1h}) \to \sqrt{\int_0^\tau \|B_h^* \dot{w}_h(t)\|_U^2 dt}$ is a norm in the finite dimensional space V_h^2 which is a consequence of our hypothesis (3.14). Now, the argument used in [18, Theorem 1.3] allows us to conclude that inequality (3.16) holds and the constant M may be chosen independent of h. □

Remark 3.4. *Contrary to the constant M in* (3.16) *which may be chosen independent of h, in general, the constant ω cannot be uniform in h, i. e.,*

$$\lim_{h \to 0} \omega(h) = 0. \tag{3.19}$$

The interested reader is referred to [24, 25] for a few interesting examples.

We have the following discrete version of Theorem 2.3, which shows the existence of a periodic solution of (3.9) by using (3.16).

Theorem 3.5. *Let $h > 0$. Assume that* (3.14) *holds and that $f_h \in \mathcal{C}([0,\infty); V_h)$ satisfies (2.14). There exists a unique $\widehat{U}_h^0 \in V_h^2$ such that the corresponding solution $\widehat{U}_h \in \mathcal{C}^1\left([0,\infty); V_h^2\right)$ of* (3.9) *with initial data $\widehat{U}_h^0$ is $T-$periodic.*

Proof. We use the same fixed point argument as in the proof of Theorem 2.3. We define the operator $\Lambda_h : V_h^2 \to V_h^2$ by

$$\Lambda_h U_h^0 = \mathbb{S}_h(T) U_h^0 + \int_0^T \mathbb{S}_h(T-s) F_h(s)\, ds, \tag{3.20}$$

where $\{\mathbb{S}_h(t)\}_{t \geqslant 0}$ is the semigroup generated by $\mathbb{A}_h$ in X_h and $U_h^0 = \begin{bmatrix} u_{0h} \\ v_{0h} \end{bmatrix}$. If $U_h = \begin{bmatrix} u_h \\ \dot{u}_h \end{bmatrix}$, from (3.16) we obtain that

$$\|\mathbb{S}_h(T) U_h^0\|_{X_h} = \left(\|A_h^{\frac{1}{2}} u_h(T)\|^2 + \|\dot{u}_h(T)\|^2 \right)^{\frac{1}{2}} \leqslant M\, e^{-\omega(h)T} \|U_h^0\|_{X_h} \tag{3.21}$$

Now, the remaining part of the proof is analogous to Theorem 2.3. □

Remark 3.6. *Hypothesis* (3.14) *(or, equivalently,* (3.15)*) ensures the exponential decay* (3.16) *of the energy and the contractive properties of the operator* Λ_h *in* V_h^2*. If* (3.14) *does not hold, equation* (3.9) *may not have periodic solution. Indeed, if* φ_{hn} *is an eigenvector of* A_h *with the property that* $B_h^*\varphi_{hn} = 0$ *and* $f_h = e^{i\frac{2\pi}{T}t}\varphi_{hn}$ *with* $T = \frac{2\pi}{\sqrt{\lambda_{hn}}}$*, then* (3.9) *has no periodic solution.*

On the other hand, the fact that the decay rate $\omega(h)$ *is not uniform with respect to* h *affects the contractive properties of* Λ_h *which may become weaker as* h *goes to zero. As we shall see in the following sections, this fact has important consequences in the periodic solution's numerical approximation process.*

4 Discrete problem: second method

In order to ensure uniform (in h) exponential decay rate in (3.16), we consider the following alternative discretization of (2.8) which adds a vanishing viscosity term

$$\begin{cases} \ddot{u}_h(t) + A_h u_h(t) + B_h B_h^* \dot{u}_h(t) + \vartheta h^\eta A_h \dot{u}_h(t) = f_h(t) \qquad (t > 0), \\ u_h(0) = u_{0h}, \ \dot{u}_h(0) = u_{1h}, \end{cases} \tag{4.1}$$

where $\vartheta \in [0, 1]$ and $\eta > 0$ is a parameter which will be chosen later on. Obviously, if $\vartheta = 0$ system (4.1) is nothing else than (3.8). The energy $E_h(t)$ of the solutions of (4.1) is defined like in (3.13) and, if $f_h \equiv 0$, verifies

$$\frac{dE_h}{dt}(t) = -\|B^*\dot{u}_h(t)\|_U^2 - \vartheta h^\eta \|A_h^{\frac{1}{2}}\dot{u}_h\|^2 \leqslant 0 \qquad (t \geqslant 0).$$

Hence, if $\vartheta > 0$, the term $\vartheta h^\eta A_h \dot{u}_h(t)$ is a numerical viscosity devoted to vanish as h tends to zero. Its purpose is to reinforces the dissipative properties of the system and, eventually, to restore the uniform decay properties of the energy. The following result may be found in [12, Theorem 7.1] (see, also, [21] for a less general case).

Theorem 4.1. *Under the above hypothesis let* $\vartheta > 0$ *and* $\eta = \theta$*. Then system* (4.1) *has a uniform exponential decay, i. e., if* $f_h \equiv 0$*, there exist two positive constants* M *and* ω *such that, for every* $h < h^*$*,*

$$E_h(t) \leqslant M^2 E_h(0) e^{-2\omega t} \qquad (t \geqslant 0 \qquad (u_{0h}, u_{1h}) \in V_h \times V_h). \tag{4.2}$$

Remark 4.2. *Note that Theorem 4.1 does not need the hypothesis* (3.14) *in order to guarantee the decay of the energy. Indeed, if* $\vartheta > 0$, (3.14) *follows immediately from the stronger dissipative properties of equation* (4.1)*. Also, note that unlike in* (3.16)*, the constant* ω *is independent of* h*.*

We can write (4.1) in the following vectorial form

$$\dot{U}_h(t) = \mathbb{A}_{h\vartheta} U_h(t) + F_h(t), \qquad (t > 0) \tag{4.3}$$

$$U_h(0) = U_h^0, \tag{4.4}$$

where

$$U_h(t) = \begin{bmatrix} u_h(t) \\ v_h(t) \end{bmatrix}, \ \mathbb{A}_{h\vartheta} = \begin{bmatrix} 0 & I \\ -A_h & -B_h B_h^* - \vartheta h^\eta A_h \end{bmatrix}, \ F_h(t) = \begin{bmatrix} 0 \\ f_h(t) \end{bmatrix} \text{ and } U_h^0 = \begin{bmatrix} u_{0h} \\ u_{1h} \end{bmatrix}.$$

Observe that $\mathbb{A}_{h\vartheta} \in \mathcal{L}(X_h)$. Therefore, $\mathbb{A}_{h\vartheta}$ is the infinitesimal generator of a uniformly continuous semigroup of linear bounded operators $\mathbb{S}_{h\vartheta}(t) = e^{t\mathbb{A}_{h\vartheta}}$ on X_h. Moreover, since $\mathbb{A}_{h\vartheta}$ is m-dissipative, the semigroup $\mathbb{S}_{h\vartheta}$ is of contractions in X_h. We have the following result concerning the existence and uniqueness of periodic solutions of (4.3).

Theorem 4.3. *Let $h \in (0, h^*)$, $\vartheta > 0$ and $\eta = \theta$. Furthermore, assume that $f_h \in \mathcal{C}([0,\infty); V_h)$ satisfies (2.14). Then there exists a unique $\widehat{U}_h^0 \in V_h \times V_h$ such that the corresponding solution $\widehat{U}_h \in \mathcal{C}^1([0,\infty); V_h \times V_h)$ of (4.3) with initial data $\widehat{U}_h^0$ is $T-$periodic.*

Proof. The proof is similar to the one of Theorem 3.5 and it is based on the contraction properties of the operator $\Lambda_{h\vartheta} : V_h^2 \to V_h^2$ defined by

$$\Lambda_{h\vartheta} U_h^0 = \mathbb{S}_{h\vartheta}(T) U_h^0 + \int_0^T \mathbb{S}_{h\vartheta}(T-s) F_h(s)\, ds. \tag{4.5}$$

□

Remark 4.4. *According to Theorem 4.1, if $\vartheta > 0$ and $\eta = \theta$, the contractive properties of $\Lambda_{h\vartheta}$ do not depend on h and the approximation process of the periodic solutions will be more efficient than in the case $\vartheta = 0$. However, in order to give a precise meaning to this statement we need to analyze conjointly the error in the approximation of* (2.13) *with* (4.1) *and the error in the approximation of the fixed point of $\Lambda_{h\vartheta}$. This analysis will provide an error estimate for the entire approximation process.*

5 Error estimate for nonhomogeneous hyperbolic problems

The aim of this section is to provide error estimates for the approximation of the solution of (2.13) with the solution of (4.1). We use the notation in Section 3 for the families of spaces $(V_h)_{h>0}$, $(U_h)_{h>0}$ and the families of operators $(\pi_h)_{h>0}$, $(\tilde{\pi}_h)_{h>0}$, $(A_h)_{h>0}$, $(B_h)_{h>0}$. Moreover, we want to give the optimal choice of the exponent $\eta > 0$ in order to have the maximum dissipative effect without affecting the convergence rate.

In the case in which $B = 0$, $\vartheta = 0$ and A is the Dirichlet Laplacian, it has been shown in Baker [3] that, given $w_0 \in H_{\frac{3}{2}}$, $w_1 \in H_1$, the solution of (4.1) converges when h tends to zero to the solution of (2.13). Moreover, [3] contains precise estimates of the convergence rate. The result below shows that the same error estimates hold when A is an arbitrary positive operator, $B \neq 0$, $\vartheta \neq 0$ and $\eta \geqslant \theta$. Throughout this section we assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$.

Proposition 5.1. *Let $w_0 \in H_{\frac{3}{2}}, w_1 \in H_1$, $f \in W^{1,1}(0,T; H_{\frac{1}{2}})$, $\vartheta \in [0,1]$. Let w and w_h be the corresponding solutions of* (2.13) *and* (4.1)*, respectively. Moreover, assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$, $\eta \geqslant \theta$, $(w_h^0, w_h^1) = (\pi_h w^0, \pi_h w^1)$ and $f_h(t) = \pi_h f(t)$ for $t \geqslant 0$. Then there exist two constants K_0, $K_1 > 0$ such that, for every $h \in (0, h^*)$, we have that*

$$\|\dot{w}(t) - \dot{w}_h(t)\| + \|w(t) - w_h(t)\|_{\frac{1}{2}} \leqslant$$
$$(K_0 + K_1 T) h^\theta \left(\|w_0\|_{\frac{3}{2}} + \|w_1\|_1 + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right) \qquad (t \in [0,T]). \tag{5.1}$$

Proof. We first note that, according to Proposition 2.2, we have

$$w \in \mathcal{C}([0,\infty); H_{\frac{3}{2}}) \cap \mathcal{C}^1([0,\infty); H_1) \cap \mathcal{C}^2([0,\infty); H_{\frac{1}{2}}),$$

$$\|\ddot{w}(t)\|_{\frac{1}{2}} + \|\dot{w}(t)\|_1 + \|w(t)\|_{\frac{3}{2}} \leqslant C\left(\|w_1\|_1 + \|w_0\|_{\frac{3}{2}} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right) \qquad (t \in [0,T]). \tag{5.2}$$

Equation (2.13) can be written

$$\langle \ddot{w}, \varphi\rangle + \langle A^{\frac{1}{2}} w, A^{\frac{1}{2}}\varphi\rangle + \langle B^* \dot{w}, B^*\varphi\rangle_U = \langle f, \varphi\rangle \qquad (\varphi \in H_{\frac{1}{2}}),$$

whereas, using (3.3) and (3.7), we see that (4.1) is equivalent to

$$\langle \ddot{w}_h, \varphi_h\rangle + \langle A^{\frac{1}{2}} w_h, A^{\frac{1}{2}}\varphi_h\rangle + \langle B^* \dot{w}_h, B^*\varphi_h\rangle_U + \vartheta h^\eta \langle A^{\frac{1}{2}} \dot{w}_h, A^{\frac{1}{2}}\varphi_h\rangle = \langle f_h, \varphi_h\rangle \qquad (\varphi_h \in V_h).$$

Taking $\varphi = \varphi_h$ in the first one of the above relations and subtracting side by side it follows that

$$\begin{aligned}\langle \ddot{w} - \ddot{w}_h, \varphi_h\rangle + \langle A^{\frac{1}{2}}(w - w_h), A^{\frac{1}{2}}\varphi_h\rangle + \langle B^*\dot{w} - B^*\dot{w}_h, B^*\varphi_h\rangle_U \\ = \vartheta h^\eta \langle A^{\frac{1}{2}}\dot{w}_h, A^{\frac{1}{2}}\varphi_h\rangle + \langle f - f_h, \varphi_h\rangle \qquad (\varphi_h \in V_h),\end{aligned} \tag{5.3}$$

which yields (recall that π_h is the orthogonal projector from $H_{\frac{1}{2}}$ onto V_h) that

$$\begin{aligned}&\langle \pi_h\ddot{w} - \ddot{w}_h, \varphi_h\rangle + \langle A^{\frac{1}{2}}(\pi_h w - w_h), A^{\frac{1}{2}}\varphi_h\rangle + \vartheta h^\eta \langle A^{\frac{1}{2}}(\pi_h\dot{w} - \dot{w}_h), A^{\frac{1}{2}}\varphi_h\rangle \\ &= \langle \pi_h\ddot{w} - \ddot{w}, \varphi_h\rangle - \langle B^*\dot{w} - B^*\dot{w}_h, B^*\varphi_h\rangle_U + \vartheta h^\eta\langle A^{\frac{1}{2}}\dot{w}, A^{\frac{1}{2}}\varphi_h\rangle + \langle f - f_h, \varphi_h\rangle \qquad (\varphi_h \in V_h).\end{aligned} \tag{5.4}$$

We set

$$\mathcal{E}_h(t) = \frac{1}{2}\|\pi_h\dot{w} - \dot{w}_h\|^2 + \frac{1}{2}\|A^{\frac{1}{2}}(\pi_h w - w_h)\|^2.$$

Using (5.4) it follows that

$$\begin{aligned}&\dot{\mathcal{E}}_h(t) + \vartheta h^\eta \|A^{\frac{1}{2}}(\pi_h\dot{w} - \dot{w}_h)\|^2 \\ &\qquad = \Re\big(\langle \pi_h\ddot{w} - \ddot{w}, \pi_h\dot{w} - \dot{w}_h\rangle - \langle B^*(\dot{w} - \dot{w}_h), B^*(\pi_h\dot{w} - \dot{w}_h)\rangle_U \\ &\qquad\qquad + \vartheta h^\eta \langle A^{\frac{1}{2}}\dot{w}, A^{\frac{1}{2}}(\pi_h\dot{w} - \dot{w}_h)\rangle + \langle f - f_h, \pi_h\dot{w} - \dot{w}_h\rangle\big) \\ &\quad = \Re\big(\langle \pi_h\ddot{w} - \ddot{w}, \pi_h\dot{w} - \dot{w}_h\rangle - \|B^*(\pi_h\dot{w} - \dot{w}_h)\|_U^2 + \langle BB^*(\pi_h\dot{w} - \dot{w}), (\pi_h\dot{w} - \dot{w}_h)\rangle \\ &\qquad\qquad + \vartheta h^\eta\langle A^{\frac{1}{2}}\dot{w}, A^{\frac{1}{2}}(\pi_h\dot{w} - \dot{w}_h)\rangle + \langle f - f_h, \pi_h\dot{w} - \dot{w}_h\rangle\big).\end{aligned}$$

We have thus shown that

$$\dot{\mathcal{E}}_h(t) \leqslant M\left(\|\pi_h\ddot{w} - \ddot{w}\| + \|\pi_h\dot{w} - \dot{w}\| + \vartheta h^\eta\|A\dot{w}\| + \|f - f_h\|\right)\|\pi_h\dot{w} - \dot{w}_h\|,$$

where $M = 1 + \|BB^*\|_{\mathcal{L}(H)}$. It follows that

$$2\,\mathcal{E}_h^{\frac{1}{2}}(t)\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant M\sqrt{2}\left(\|\pi_h\ddot{w} - \ddot{w}\| + \|\pi_h\dot{w} - \dot{w}\| + \vartheta h^\eta\|A\dot{w}\| + \|f - f_h\|\right)\mathcal{E}_h^{\frac{1}{2}}(t),$$

which yields

$$\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant \mathcal{E}_h^{\frac{1}{2}}(0)+\frac{M}{\sqrt{2}}\int_0^t \left(\|\pi_h\ddot{w}-\ddot{w}\|+\|\pi_h\dot{w}-\dot{w}\|+\vartheta h^\eta\|A\dot{w}\|+\|f-f_h\|\right)\,\mathrm{d}s \qquad (t\geqslant 0).$$

The above estimate, combined with (5.2), to the fact that $\mathcal{E}_h(0)=0$ and to (3.2), implies that there exists a constant $\widetilde{K}_1>0$ such that, for every $h\in(0,h^*)$, we have

$$\mathcal{E}_h^{\frac{1}{2}}(t) \leqslant t\,\widetilde{K}_1\left(h^\theta+\vartheta h^\eta\right)\left(\|w_0\|_{\frac{3}{2}}+\|w_1\|_1+\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right) \qquad (t\in[0,T]). \tag{5.5}$$

On the other hand, using (5.2), combined with (3.1) and (3.2), we have that, for every $h\in(0,h^*)$,

$$\begin{aligned}\|\dot{w}(t)-\dot{w}_h(t)\| &\leqslant \|\dot{w}(t)-\pi_h\dot{w}(t)\|+\|\pi_h\dot{w}(t)-\dot{w}_h(t)\|\\ &\leqslant K_0\left[h^\theta\left(\|w_0\|_{\frac{3}{2}}+\|w_1\|_1+\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right)+\mathcal{E}_h^{\frac{1}{2}}(t)\right],\end{aligned}$$

$$\begin{aligned}&\|w(t)-w_h(t)\|_{\frac{1}{2}} \leqslant \|w(t)-\pi_h w(t)\|_{\frac{1}{2}}+\|\pi_h w(t)-w_h(t)\|_{\frac{1}{2}}\\ &\leqslant K_0\left[h^\theta\left(\|w_0\|_{\frac{3}{2}}+\|w_1\|_1+\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right)+\mathcal{E}_h^{\frac{1}{2}}(t)\right],\end{aligned}$$

for some constant $K_0>0$. The last two inequalities, combined with (5.5) and taking into account that $\eta\geqslant\theta$, yield the conclusion (5.1). ☐

6 Error estimates for periodic solutions

In this section we study the convergence and we provide error estimates for the approximation of the periodic solution of (2.13). Let Λ and $\Lambda_{h\vartheta}$ be the operators from Theorems 2.3 and 4.3, respectively, defined as follows

$$\Lambda : H_{\frac{1}{2}}\times H \to H_{\frac{1}{2}}\times H, \qquad U^0\to\Lambda(U^0)=U(T),$$

$$\Lambda_{h\vartheta} : V_h\times V_h\to V_h\times V_h, \qquad U_h^0\to\Lambda_{h\vartheta}(U_h^0)=U_h(T),$$

where U and U_h are the solutions of (2.15)-(2.16) and (4.3)-(4.4), respectively. Λ_{h0} will be denoted by Λ_h. According to Proposition 3.3 and Theorem 4.1, the discrete semigroup $\{\mathbb{S}_{h\vartheta}(t)\}_{t\geqslant 0}$ introduced in Section 4 verifies

$$\|\mathbb{S}_{h\vartheta}(t)U_h^0\|_X \leqslant Me^{-\omega(h,\vartheta)t}\|U_h^0\|_X \qquad (t\geqslant 0, \qquad U_h^0\in V_h\times V_h), \tag{6.1}$$

where M is a positive constant independent of h and $\omega(h,\vartheta)$ is a positive constant which depends on h if $\vartheta=0$ and it is independent of h when $\vartheta>0$. We recall that the norm in the space $V_h\times V_h$ is the one induced from $H_{\frac{1}{2}}\times H$. Moreover, we suppose that the continuous semigroup $\{\mathbb{S}(t)\}_{t\geqslant 0}$ verifies (2.18).

Let $\widehat{U}^0$ be the fixed point of Λ and $\widehat{U}_h^0$ be the fixed point of $\Lambda_{h\vartheta}$. Given U_h^0, the aim of this section is to estimate the differences $\widehat{U}^0-\Lambda_{h\vartheta}^n U_h^0$ for each $n\geqslant 1$. As a consequence, an estimate for $\widehat{U}^0-\widehat{U}_h^0$ will be also obtained in the case $\vartheta>0$.

Remark 6.1. *In this paper we approximate the fixed point $\widehat{U}^0$ of Λ by iterating the operator $\Lambda_{h\vartheta}$. Variational methods are considered in [5, 9, 14, 29]. Although the error estimates in those cases may be different, the qualitative results and phenomena are similar.*

Let $q = e^{-\omega T}$, $q_h = e^{-\omega(h,\vartheta)T}$ when $\vartheta = 0$ and $r = e^{-\omega(h,\vartheta)T}$ when $\vartheta > 0$. As we have mentioned before, q_h depends on h and it may tend to 1 as h goes to zero while r is independent of h.

Note that (2.18) and (6.1) imply that $\frac{1}{M}\Lambda$ is a $q-$contraction in $H_{\frac{1}{2}} \times H$ and $\frac{1}{M}\Lambda_{h\vartheta}$ is a q_h-contraction in $V_h \times V_h$. Thus we cannot apply directly to the operators Λ and $\Lambda_{h\vartheta}$ the well-known estimates in the fixed point approximation algorithm (see, for instance, [7, Theorem 5.1.1]). However, we have the following similar result.

Proposition 6.2. *Let $\widehat{U}^0$ be the unique fixed point of Λ. For any $U^0 \in X$, the sequence $(\Lambda^n U^0)_{n\geqslant 0}$ converges to $\widehat{U}^0$ and the following estimates hold*

$$\begin{aligned}
&\|\widehat{U}^0 - \Lambda^n U^0\|_X \leqslant Mq^n \|\widehat{U}^0 - U^0\|_X \qquad (n \geqslant 0),\\
&\|\widehat{U}^0 - \Lambda^n U^0\|_X \leqslant \frac{Mq^n}{1-q}\|\Lambda U^0 - U^0\|_X \qquad (n \geqslant 0),\\
&\|\widehat{U}^0 - \Lambda^n U^0\|_X \leqslant \frac{Mq}{1-q}\|\Lambda^n U^0 - \Lambda^{n-1}U^0\|_X \qquad (n \geqslant 1).
\end{aligned} \tag{6.2}$$

Proof. For the first inequality in (6.2), note that

$$\|\widehat{U}^0 - \Lambda^n U^0\|_X = \|\Lambda^n \widehat{U}^0 - \Lambda^n U^0\|_X = \|\mathbb{S}(nT)(\widehat{U}^0 - U^0)\|_X \leqslant Mq^n \|\widehat{U}^0 - U^0\|_X.$$

For the second inequality remark first that

$$\|\Lambda^{k+1}U^0 - \Lambda^k U^0\|_X = \|\Lambda^k \Lambda U^0 - \Lambda^k U^0\|_X \leqslant Mq^k \|\Lambda U^0 - U^0\|_X \qquad (k \geqslant 0). \tag{6.3}$$

By using (6.3) we deduce that

$$\begin{aligned}
\|\Lambda^{n+k+1}U^0 - \Lambda^n U^0\|_X \leqslant \sum_{j=0}^{k} \|\Lambda^{n+k+1-j}U^0 - \Lambda^{n+k-j}U^0\|_X \leqslant \\
M\sum_{j=0}^{k} q^{n+k-j}\|\Lambda U^0 - U^0\|_X = Mq^n \frac{1-q^{k+1}}{1-q}\|\Lambda U^0 - U^0\|_X.
\end{aligned} \tag{6.4}$$

The second inequality from (6.2) follows from (6.4) by letting k to tend to infinity. Finally, the third inequality from (6.2) is obtained by using in (6.4) the inequality

$$\|\Lambda^{n+k+1-j}U^0 - \Lambda^{n+k-j}U^0\|_X \leqslant Mq^{k+1-j}\|\Lambda^n U^0 - \Lambda^{n-1}U^0\|_X.$$

$\square$

Also, we have the following result.

Proposition 6.3. *Let $\vartheta \in [0,1]$ and $\widehat{U}_h^0$ be the unique fixed point of $\Lambda_{h\vartheta}$. For any $U_h^0 \in V_h \times V_h$, the sequence $(\Lambda_{h\vartheta}^n U_h^0)_{n\geqslant 0}$ converges to $\widehat{U}_h^0$ and the following estimates hold*

$$\begin{aligned}
&\|\widehat{U}_h^0 - \Lambda_{h\vartheta}^n U_h^0\|_{V_h^2} \leqslant M q_h^n \|\widehat{U}_h^0 - U_h^0\|_{V_h^2} \qquad (n \geqslant 0),\\
&\|\widehat{U}_h^0 - \Lambda_{h\vartheta}^n U_h^0\|_{V_h^2} \leqslant \frac{M q_h^n}{1-q_h} \|\Lambda_{h\vartheta} U_h^0 - U_h^0\|_{V_h^2} \qquad (n \geqslant 0),\\
&\|\widehat{U}_h^0 - \Lambda_{h\vartheta}^n U_h^0\|_{V_h^2} \leqslant \frac{M q_h}{1-q_h} \|\Lambda_{h\vartheta}^n U_h^0 - \Lambda_{h\vartheta}^{n-1} U_h^0\|_{V_h^2} \qquad (n \geqslant 1).
\end{aligned} \tag{6.5}$$

Proof. It is similar to the proof of Proposition 6.2 and we omit it. ☐

We pass now to estimate the error in the approximation of $\widehat{U}^0$. In the sequel Π_h will denote the projection operator $\begin{bmatrix} \pi_h & 0 \\ 0 & \pi_h \end{bmatrix}$. We have the following result.

Theorem 6.4. *Let f be a function which verifies (2.14) and $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. Assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and (3.14) holds. Let $\widehat{U}^0$ be the unique fixed point of Λ given by Theorem 2.3. By taking $f_h = \pi_h f$, let $\Lambda_h = \Lambda_{h0}$ be the discrete operator from Theorem 3.5. Then there exists a constant $C > 0$ such that, for each $U^0 \in X_1$, $n \geqslant 1$ and $h < h^*$, the following estimate holds*

$$\|\widehat{U}^0 - \Lambda_h^n(\Pi_h U^0)\|_X \leqslant C \left(n\, h^\theta + \frac{q^n}{1-q} \right) \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \tag{6.6}$$

Proof. Let $U^0 \in X_1 = H_{\frac{3}{2}} \times H_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$. From (6.2) we deduce that

$$\begin{aligned}
\|\widehat{U}^0 - \Lambda_h^n(U_h^0)\|_X \leqslant \|\widehat{U}^0 - \Lambda^n U^0\|_X + \|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X \leqslant \\
\frac{M q^n}{1-q} \|\Lambda U^0 - U^0\|_X + \|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X.
\end{aligned} \tag{6.7}$$

From (6.7) and by taking into account that

$$\|\Lambda U^0 - U^0\|_X \leqslant C \left[\|U^0\|_X + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right],$$

(6.6) will follow after estimating the difference $\|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X$. By using (6.1), we deduce that

$$\begin{aligned}
\|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X = \|\Lambda^n U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant \\
\leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h \Pi_h \Lambda^{n-1} U^0\|_X + \\
+ \|\Lambda_h \Pi_h \Lambda^{n-1} U^0 - \Lambda_h^2 \Pi_h \Lambda^{n-2} U^0\|_X + \|\Lambda_h^2 \Pi_h \Lambda^{n-2} U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant \\
\leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h \Pi_h \Lambda^{n-1} U^0\|_X + \\
+ M q_h \|\Pi_h \Lambda^{n-1} U^0 - \Lambda_h \Pi_h \Lambda^{n-2} U^0\|_X + \|\Lambda_h^2 \Pi_h \Lambda^{n-2} U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant \\
\leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h \Pi_h \Lambda^{n-1} U^0\|_X + M q_h \|\Pi_h \Lambda^{n-1} U^0 - \Lambda_h \Pi_h \Lambda^{n-2} U^0\|_X \\
+ \|\Lambda_h^2 \Pi_h \Lambda^{n-2} U^0 - \Lambda_h^3 \Pi_h \Lambda^{n-3} U^0\|_X + \|\Lambda_h^3 \Pi_h \Lambda^{n-3} U^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant
\end{aligned}$$

$$\leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \|\Pi_h \Lambda^n U^0 - \Lambda_h \Pi_h \Lambda^{n-1} U^0\|_X + M q_h \|\Pi_h \Lambda^{n-1} U^0 - \Lambda_h \Pi_h \Lambda^{n-2} U^0\|_X$$
$$+ M q_h^2 \|\Pi_h \Lambda^{n-2} U^0 - \Lambda_h \Pi_h \Lambda^{n-3} U^0\|_X + \|\Lambda_h^3 \Pi_h \Lambda^{n-3} U^0 - \Lambda_h^n \Pi_h U^0\|_X.$$

We repeat this argument and obtain that

$$\|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X \leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + \\ + M \sum_{j=0}^{n-1} q_h^j \|\Pi_h \Lambda^{n-j} U^0 - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X. \quad (6.8)$$

Since $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$, $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and $U^0 \in X_1$, we can use Proposition 5.1 (with $\vartheta = 0$) to estimate $\|\Pi_h \Lambda(\Lambda^{n-1-j} U^0) - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X$. Thus, from (3.1), (3.2) and Proposition 5.1 we deduce that, for any $h < h^*$,

$$\|\Pi_h \Lambda^{n-j} U^0 - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X = \|\Pi_h \Lambda(\Lambda^{n-1-j} U^0) - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X \leqslant \\ \|\Pi_h \Lambda(\Lambda^{n-1-j} U^0) - \Lambda(\Lambda^{n-1-j} U^0)\|_X + \|\Lambda(\Lambda^{n-1-j} U^0) - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X \leqslant \\ C h^\theta \|\Lambda^{n-j} U^0\|_{X_1} + (K_0 + K_1 T) h^\theta \left(\|\Lambda^{n-1-j} U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right).$$

Since the sequence $(\Lambda^m U^0)_{m \geqslant 0}$ verifies the estimate

$$\|\Lambda^m U^0\|_{X_1} \leqslant C \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right) \qquad (m \geqslant 0), \quad (6.9)$$

it follows that there exists a constant $C = C(T)$ such that, for each $j \geqslant 0$ and $h < h^*$,

$$\|\Pi_h \Lambda(\Lambda^{n-1-j} U^0) - \Lambda_h \Pi_h \Lambda^{n-1-j} U^0\|_X \leqslant C h^\theta \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \quad (6.10)$$

Also, from (3.1), (3.2) and Proposition 5.1 we have that, for any $n \geqslant 0$ and $h < h^*$,

$$\|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X \leqslant C h^\theta \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \quad (6.11)$$

Since $q_h < 1$, from (6.8), (6.10) and (6.11) we deduce that, for any $n \geqslant 0$ and $h < h^*$,

$$\|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X \leqslant C h^\theta \left(1 + M \sum_{j=0}^{n-1} q_h^j \right) \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right) \leqslant \\ \leqslant C h^\theta \, n \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \quad (6.12)$$

From (6.7) and (6.12) we conclude that (6.6) holds and the proof ends. □

Remark 6.5. *By choosing* $n = \left[\frac{\theta}{|\ln(q)|} \ln\left(\frac{1}{h}\right) \right] + 1$ *(for each real number* a*,* $[a]$ *denotes the largest integer smaller than* a*), we deduce from (6.6) that the following estimate holds for each* $U^0 \in X_1$

$$\|\widehat{U}^0 - \Lambda_h^n \Pi_h U^0\|_X \leqslant C h^\theta \ln\left(\frac{1}{h}\right) \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \right). \quad (6.13)$$

Note that the error in the approximation of $\widehat{U}^0$ *is slightly larger than the one of the Galerkin scheme,* h^θ*, by the factor* $\ln\left(\frac{1}{h}\right)$*.*

As a consequence of Theorem 6.4 an estimate for the difference between the fixed points of Λ and Λ_h may be given. We have the following result.

Corollary 6.6. *Let f be a function which verifies (2.14) and $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. Assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and (3.14) hold. By taking $f_h = \pi_h f$, let $\widehat{U}^0$ and $\widehat{U}_h^0$ be the unique fixed points of Λ and Λ_h given by Theorems 2.3 and 3.5, respectively. Then there exists a constant $C > 0$ such that, for each $n \geqslant 1$ and $h < h^*$, the following estimate holds*

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant C\left(n\, h^\theta + \frac{q^n}{1-q} + \frac{q_h^n}{1-q_h}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}. \tag{6.14}$$

Proof. Let $U^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in X_1$, $U_h^0 = \Pi_h U^0$ and $n \geqslant 1$. From (6.6) and (6.5) we deduce that

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant \|\widehat{U}^0 - \Lambda_h^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X \leqslant$$
$$C\left(n\, h^\theta + \frac{q^n}{1-q}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + \frac{M q_h^n}{1-q_h}\|\Lambda_h U_h^0 - U_h^0\|_{V_h^2}. \tag{6.15}$$

Since

$$\|\Lambda_h U_h^0 - U_h^0\|_X \leqslant C\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}$$

estimate (6.14) follows from (6.15) and the proof ends. □

Remark 6.7. *Note that (6.14) does not ensure the convergence of $(\widehat{U}_h^0)_{h>0}$ to $\widehat{U}^0$ as h goes to zero. Indeed, since $q_h = e^{-\omega(h)T}$ may tend to 1 as h goes to zero (see Remark 3.4), we cannot guarantee that the terms nh^θ and $\frac{q_h^n}{1-q_h}$ are simultaneously tending to zero as h does. As we shall see, this phenomenon does not occur if a vanishing viscosity is introduced and (4.1) is used with $\vartheta > 0$. Also, a better error estimate than (6.13) will be obtained in this case, as Theorem 6.8 below proves.*

Now, let us pass to study the discretization method introduced in Section 4 in which a numerical vanishing viscosity is introduced. Our first result concerns the error estimate between $\widehat{U}^0$ and the iterates of the operator $\Lambda_{h\vartheta}$.

Theorem 6.8. *Let f be a function which verifies (2.14) and $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. Assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and let $\widehat{U}^0$ be the unique fixed point of Λ given by Theorem 2.3. By taking $f_h = \pi_h f$, let $\Lambda_{h\vartheta}$ be the discrete operator from Theorem 4.3. If $\vartheta > 0$, there exists a constant $C > 0$ such that, for each $U^0 \in X_1$, $n \geqslant 1$ and $h < h^*$, the following estimate holds*

$$\|\widehat{U}^0 - \Lambda_{h\vartheta}^n \Pi_h U^0\|_X \leqslant C\left(h^\theta + \frac{q^n}{1-q}\right)\left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right). \tag{6.16}$$

Proof. The only major difference with respect to the proof of Theorem 6.4 is that q_h has to be replaced by r and estimate (6.12) becomes

$$\|\Lambda^n U^0 - \Lambda^n_{\vartheta h}\Pi_h U^0\|_X \leqslant \|\Lambda^n U^0 - \Pi_h \Lambda^n U^0\|_X + M\sum_{j=0}^{n-1} r^j \|\Pi_h \Lambda^{n-j}U^0 - \Lambda_{\vartheta h}\Pi_h \Lambda^{n-1-j}U^0\|_X \leqslant$$
$$\leqslant Ch^\theta \left(1 + M\sum_{j=0}^{n-1} r^j\right)\left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right) \leqslant$$
$$\leqslant Ch^\theta \left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right) \quad (6.17)$$

By using (6.17) in (6.7), inequality (6.16) is proved. □

Also, we have the following result concerning the convergence of the family $(\widehat{U}^0_h)_{h>0}$ of discrete fixed points of $\Lambda_{\vartheta h}$.

Corollary 6.9. *Let f be a function which verifies* (2.14) *and $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. Assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$, $\vartheta > 0$ and $\eta = \theta$. Let $\widehat{U}^0$ and $\widehat{U}^0_h$ be the unique fixed points of Λ and $\Lambda_{h\vartheta}$ given by Theorems 2.3 and 4.3, respectively. Then there exists a constant $C > 0$ such that, for each $n \geqslant 1$ and $h < h^*$, the following estimate holds*

$$\|\widehat{U}^0 - \widehat{U}^0_h\|_X \leqslant C\left(h^\theta + \frac{q^n}{1-q} + \frac{r^n}{1-r}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}. \quad (6.18)$$

Proof. It is similar to that of Corollary 6.6, but using (6.16) instead of (6.6). □

Remark 6.10. *Unlike* (6.14) *from Corollary 6.6, estimate* (6.18) *ensures the convergence of $\widehat{U}^0_h$ to $\widehat{U}^0$ as h tends to zero and allows us to determine a number n of iterations needed to guarantee a uniform error estimate. Indeed, by taking $n = \left[\frac{\theta}{|\ln(\max\{q,r\})|}\ln\left(\frac{1}{h}\right)\right] + 1$, we deduce from* (6.18) *that the following estimate holds*

$$\|\widehat{U}^0 - \widehat{U}^0_h\|_X \leqslant Ch^\theta \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \qquad (n \geqslant 1). \quad (6.19)$$

Moreover, by choosing $n = \left[\frac{\theta}{|\ln(q)|}\ln\left(\frac{1}{h}\right)\right] + 1$, we deduce from (6.16) *that*

$$\|\widehat{U}^0 - \Lambda^n_{\vartheta h}\Pi_h U^0\|_X \leqslant Ch^\theta\left(\|U^0\|_{X_1} + \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})}\right) \qquad (U^0 \in X_1, \ \ n \geqslant 1). \quad (6.20)$$

Note that the error estimate (6.20)*, when compared to* (6.13)*, indicates a faster convergence of the scheme in the presence of viscosity.*

7 Monochromatic forcing terms

In this section we suppose that the nonhomogeneous periodic term f from (2.13) has the following particular form

$$f(t,x) = e^{i\varsigma t}g(x), \quad (7.1)$$

where $\varsigma \in \mathbb{R}$ and $g \in H$. Evidently, functions of the form (7.1) are periodic of period $T = \frac{2\pi}{\varsigma}$ and are usually called *monochromatic*. They appear in many important applications including acoustic, electromagnetic and geophysical wave propagation. For instance, the wave equation

$$u_{tt}(t,x) - \Delta u(t,x) = e^{i\varsigma t} g(x), \qquad (x \in \Omega,\ t > 0) \tag{7.2}$$

has a periodic solution $u = e^{i\varsigma t} w$ if and only if w verifies the Helmholtz's equation

$$(\varsigma^2 + \Delta) w(x) = -g(x), \qquad (x \in \Omega). \tag{7.3}$$

It is well known that, for large values of ς, discretizing (7.3) leads to large indefinite linear systems. These large systems are usually solved by preconditioning and iterative method (see, for instance [6] or [23]). In [8] an alternative method to solve (7.3) was proposed and was compared positively to the more traditional methods used to solve the Helmholtz's equation (see [8][Sections 7 and 8] or [9][Section 7.6]). This approach, consisting in obtaining firstly the periodic solution of (7.2) and from it deducing the solution of (7.3), has been intensively studied in a series of papers of Glowinski et al. [8, 9, 14], Bardos and Rauch [5] and Zuazua [29]. All these papers were interested in the approximation of the outgoing solutions of Helmholtz's equation in an exterior domain $\Omega \subset \mathbb{R}^n$. For numerical reasons, the unbounded domain Ω has to be limited by introducing an artificial boundary Γ with a Sommerfeld condition

$$\frac{\partial u}{\partial n}(t,x) + u_t(t,x) = 0, \qquad (x \in \Gamma,\ t > 0). \tag{7.4}$$

Hence, (7.2)-(7.4) form a dissipative system, whose periodic solutions have to be determined. Note that, unlike (2.13) where the damping operator B is bounded, system (7.2)-(7.4) has a boundary dissipation which would correspond to an unbounded operator B. Moreover, those articles use a least square approach and design quadratic functionals for the periodically forced wave equation whose minimizers yield the solution of the Helmholtz equation one is looking for. However, as in our iterative methods, difficulties in the approximation of the periodic solutions may appear due to the fact that the exponential decay of the discretized systems is generally not uniform with respect to the mesh size.

The aim of this section is to show that, for the particular choice of the monochromatic periodic function (7.1), the lack of uniform exponential decay of the discrete semigroup is not essential and the convergence of the discrete fixed points is ensured even without the vanishing viscosity. In order to fulfil our objective, the following decomposition of the space V_h in low and high frequencies is needed.

Lemma 7.1. *Let $\varsigma \in \mathbb{R}$ be given. There exists $h_0 > 0$ with the property that, for every $h < h_0$, there exist two subspaces W_h^1 and W_h^2 of V_h such that*

1. *V_h may be written as*

$$V_h = W_h^1 \oplus W_h^2 \tag{7.5}$$

2. *There exist two positive constants M_1 and ω_1, independent of h, such that*

$$\|\mathbb{S}_h(t) U_h^0\|_X^2 \leqslant M_1 e^{-\omega_1 t} \|U_h^0\|_X \qquad (t \geqslant 0,\ U_h^0 \in W_h^1 \times W_h^1) \tag{7.6}$$

3. *There exists a constant $C > 0$, independent of h, such that*

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1} U_h^0\right\|_X \leqslant C h^\theta \|U_h^0\|_X \qquad (U_h^0 \in W_h^2 \times W_h^2). \tag{7.7}$$

Proof. We recall that $(\varphi_{hn})_{1\leqslant n\leqslant N(h)}$ and $(\lambda_{hn})_{1\leqslant n\leqslant N(h)}$ are the sets of eigenvectors normalized in V_h and eigenvalues of the operator $A_h^{\frac{1}{2}}$, respectively. Since $A_h^{\frac{1}{2}}$ is self-adjoint, $(\varphi_{hn})_{1\leqslant n\leqslant N(h)}$ forms an orthonormal basis in V_h and $(\Phi_{hn})_{1\leqslant |n|\leqslant N(h)}$ defined by (3.12) forms an orthonormal basis of V_h^2. We define the following subspaces of V_h

$$\begin{aligned} W_h^1 &= \operatorname{Span}\left\{\varphi_{hn} \mid \lambda_{hn} \leqslant \tfrac{\delta}{h^\theta}\right\}, \\ W_h^2 &= \left[W_h^1\right]^\perp, \end{aligned} \tag{7.8}$$

where $\delta > 0$ is a sufficiently small number to be chosen latter on.

We have that $V_h = W_h^1 \oplus W_h^2$. Moreover, according to [12, Theorem 8.1] there exist positive constants h_0, δ and k_T such that the following inequality holds for any $h < h_0$

$$\int_0^T \|B_h^* \dot{w}_h(t)\|_U^2 \,\mathrm{d}t \geqslant k_T \left(\|A_h^{\frac{1}{2}} w_{0h}\|^2 + \|w_{1h}\|^2 \right) \qquad ((w_{0h}, w_{1h}) \in (W_h^1)^2), \tag{7.9}$$

where w_h is the solution of the homogeneous equation

$$\begin{cases} \ddot{w}_h(t) + A_h w_h(t) = 0, & (t > 0) \\ w_h(0) = w_{0h}, \qquad \dot{w}_h(0) = w_{1h}. \end{cases}$$

From (7.9) we deduce, like in the last part of Proposition 3.3, that there exist two positive constants M_1 and ω_1, independent of h, such that (7.6) holds in $\left(W_h^1\right)^2$.

Now, let us prove (7.7). We take

$$U_h^0 = \sum_{\lambda_{h\,|n|} > \frac{\delta}{h^\theta}} a_n \Phi_{hn} \in \left(W_h^2\right)^2$$

and we remark that

$$\begin{aligned} &\left\|(i\varsigma I - \mathbb{A}_h)^{-1} U_h^0 - (i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leqslant \\ &\left\|(i\varsigma I - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)} \left\|U_h^0 - (i\varsigma I - \mathbb{A}_h^1 - \mathbb{B}_h)(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leqslant \\ &\left\|(i\varsigma I - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)} \left\|\mathbb{B}_h (i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X . \end{aligned}$$

Since the operators $\mathbb{B}_h$ and $(i\varsigma I - \mathbb{A}_h)^{-1}$ have uniformly bounded norms in h, we deduce from the above inequality that there exists a positive constant $C_1 > 0$ such that

$$\left\|(i\varsigma I - \mathbb{A}_h)^{-1} U_h^0\right\|_X \leqslant C_1 \left\|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X . \tag{7.10}$$

Remark that, at the same time, h_0 and δ can be chosen such that

$$|\varsigma| < \frac{\delta}{2h^\theta} \text{ for every } h < h_0$$

and, hence the operator $(i\varsigma I - \mathbb{A}_h^1)^{-1}$ is well defined in $\mathcal{L}((W_h^2)^2)$. Moreover, we have that

$$\left\|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X = \left\| \sum_{\Phi_{hn} \in (W_h^2)^2} \frac{a_n}{i\varsigma - i\lambda_{h\,|n|}} \Phi_{hn} \right\|_X \leqslant \max_{\Phi_{hn} \in (W_h^2)^2} \frac{1}{|\varsigma - \lambda_{h\,|n|}|} \|U_h^0\|_X .$$

Since $\Phi_{hn} \in (W_h^2)^2$ implies that $\lambda_{hn} > \frac{\delta}{h^\theta}$, we deduce that there exists a constant $C_2 > 0$ such that the following inequality holds

$$\left\|(i\varsigma I - \mathbb{A}_h^1)^{-1} U_h^0\right\|_X \leqslant C_2 h^\theta \|U_h^0\|_X \qquad \left(U_h^0 \in \left(W_h^2\right)^2,\ h < h_0\right). \tag{7.11}$$

From (7.10) and (7.11) it follows that (7.7) is verified and the proof of the Lemma ends. □

Now we can give a new estimate for the approximation of the periodic solutions of (2.13) with periodic source term f of type (7.1) by using the discrete equation (3.8) (without numerical viscosity). In the sequel we use the notation $r_1 = e^{-\omega_1 T} < 1$.

Theorem 7.2. *Let $\varsigma \in \mathbb{R}$, $T = \frac{2\pi}{\varsigma}$ and $g \in H_{\frac{1}{2}}$ be given and let $f \in W^{1,1}(0, T; H_{\frac{1}{2}})$ be defined by* (7.1)*. Assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ and that* (3.14) *holds. By taking $f_h(t) = e^{i\varsigma t}\pi_h g$, let $\widehat{U}^0$ and $\widehat{U}_h^0$ be the unique fixed points of Λ and Λ_h given by Theorems 2.3 and 3.5, respectively. Then there exist $h_1 > 0$ and $K > 0$, such that*

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant K\left(nh^\theta + \frac{q^n}{1-q} + r_1^n\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \qquad (h < h_1,\ n \geqslant 1). \tag{7.12}$$

Proof. Let $U^0 \in X_1$ such that $\Pi_h U^0 := U_h^0 \in (W_h^1)^2$. We have that

$$\Lambda_h^n U_h^0 = \mathbb{S}_h(nT)(U_h^0 - \widehat{U}_h^0) + \widehat{U}_h^0. \tag{7.13}$$

Moreover, it can be easily seen that $\widehat{U}_h^0$ is a solution of the equation

$$(i\varsigma - \mathbb{A}_h)\widehat{U}_h^0 = G_h, \tag{7.14}$$

where $G_h = \begin{bmatrix} 0 \\ \pi_h g \end{bmatrix}$.

By using Lemma 7.1 and taking $h < h_0$, we deduce that there exist two unique elements $g_h^1 \in W_h^1$ and $g_h^2 \in W_h^2$ such that $\pi_h g = g_h^1 + g_h^2$. Let us denote by $G_h^i = \begin{bmatrix} 0 \\ g_h^i \end{bmatrix}$, $i = 1, 2$. From (7.13) and (7.14), it follows that

$$\|\Lambda_h^n U_h^0 - \widehat{U}_h^0\|_X = \|\mathbb{S}_h(nT)(U_h^0 - \widehat{U}_h^0)\|_X \leqslant \left\|\mathbb{S}_h(nT)(U_h^0)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h)\right\|_X.$$

Since $U_h^0 \in (W_h^1)^2$, from (7.6) we deduce that

$$\left\|\mathbb{S}_h(nT)(U_h^0)\right\|_X \leqslant M_1 e^{-\omega_1 nT}\|U_h^0\|_X \qquad (n \geqslant 0). \tag{7.15}$$

On the other hand, by writing $G_h = G_h^1 + G_h^2$ and by using properties (7.6) and (7.7) of the spaces W_h^1 and W_h^2, we deduce that

$$\begin{aligned}\left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h)\right\|_X &\leqslant \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^1)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^2)\right\|_X = \\ &= \left\|(i\varsigma - \mathbb{A}_h)^{-1}\mathbb{S}_h(nT)(G_h^1)\right\|_X + \left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h^2)\right\|_X \leqslant \\ &\leqslant M_1 e^{-\omega_1 nT}\left\|(i\varsigma - \mathbb{A}_h)^{-1}\right\|_{\mathcal{L}(V_h)}\|G_h^1\|_X + Ch^\theta\|G_h^2\|_X.\end{aligned}$$

We obtain that there exists a constant $K > 0$ such that

$$\left\|\mathbb{S}_h(nT)(i\varsigma - \mathbb{A}_h)^{-1}(G_h)\right\|_X \leqslant K\left(e^{-\omega_1 nT} + h^\theta\right)\|G_h\|_X. \tag{7.16}$$

From (7.15) and (7.16) it follows that

$$\|\Lambda_h^n U_h^0 - \widehat{U}_h^0\|_X \leqslant K\left(r_1^n + h^\theta\right)\left(\|f\|_{L^2(0,T;H_{\frac{1}{2}})} + \|U^0\|_X\right). \tag{7.17}$$

Now, we can conclude like in Corollary 6.6. Indeed, by taking $U^0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \in X_1$, $U_h^0 = \Pi_h U^0 \in (W_h^1)^2$ and $n \geqslant 1$, we deduce from (6.2), (6.12) and (7.17) that, for any $h < h_1 := \min\{h_0, h^*\}$,

$$\begin{aligned}\|\widehat{U}^0 - \widehat{U}_h^0\|_X &\leqslant \|\widehat{U}^0 - \Lambda^n U^0\|_X + \|\Lambda^n U^0 - \Lambda_h^n U_h^0\|_X + \|\widehat{U}_h^0 - \Lambda_h^n U_h^0\|_X \leqslant \\ &\frac{Mq^n}{1-q}\|\Lambda U^0 - U^0\|_X + Cn\,h^\theta\,\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} + K\left(r_1^n + h^\theta\right)\|f\|_{L^2(0,T;H_{\frac{1}{2}})}.\end{aligned} \tag{7.18}$$

The proof of the Theorem is completed. □

Remark 7.3. *If we compare* (7.12) *with* (6.14) *we can see that, in the particular case* (7.1) *of a monochromatic source f, the convergence of the discrete fixed points of Λ_h is ensured, in spite of the fact that the contractive properties of Λ_h are not uniform in h. Hence, in this particular case, the viscosity is not needed in order to guarantee the convergence and it confirms the good numerical results obtained by Glowinski et all. in the similar* (7.2)-(7.4) *problem (see [9, 14]).*

Remark 7.4. *Results of the same type are obtained if, instead of* (7.1), *we consider*

$$f(t) = \sum_{n=1}^{p} e^{i\varsigma_n t} g_n \tag{7.19}$$

with $\varsigma_n \in \mathbb{R}$ multiples of the same number $\frac{2\pi}{T}$ and $g_n \in H_{\frac{1}{2}}$ for each $n \in \{1, 2, ..., N\}$. The important fact here is that we can still assume that there exists subspaces W_h^1 and W_h^2 of V_h such that (7.5)-(7.7) *are verified, where* (7.7) *has to be replaced by*

$$\left\|(i\,\varsigma_n\, I - \mathbb{A}_h)^{-1} U_{0h}\right\| \leqslant Ch^\theta \|U_{0h}\| \qquad (1 \leqslant n \leqslant p, \quad U_{0h} \in W_h^2 \times W_h^2). \tag{7.20}$$

Note that condition (7.20) *cannot be fulfilled if f is an arbitrary periodic function which can be written in the form*

$$f(t) = \sum_{n=1}^{\infty} e^{i\frac{2\pi n}{T}t} g_n. \tag{7.21}$$

In this general case, the filtering the high frequencies of the source term becomes mandatory. A way to do this consists in using the vanishing viscosity method studied in the previous sections. Another possibility is to directly cut-off the high frequencies of the source term. The following theorem gives a result in this direction.

Theorem 7.5. *Let be a function which verifies* (2.14) *and $f_{|[0,T]} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. Consider a sequence of discretizations $(f_h)_{h\in(0,h_1)}$ of it such that*

$$f_h(t) \in W_h^1 \qquad (t \geqslant 0, \quad h \in (0, h_1)), \tag{7.22}$$

$$\|f(t) - f_h(t)\| \leqslant Ch^\theta \|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \qquad (t \in [0,T], \quad h \in (0, h_1)). \tag{7.23}$$

Moreover, assume that $BB^* \in \mathcal{L}(H_1, H_{\frac{1}{2}})$ *and that* (3.14) *holds. Let* $\widehat{U}^0$ *and* $\widehat{U}_h^0$ *be the unique fixed points of* Λ *and* Λ_h *given by Theorems 2.3 and 3.5, respectively. Then there exist* $h_1 > 0$ *and* $K > 0$*, such that*

$$\|\widehat{U}^0 - \widehat{U}_h^0\|_X \leqslant K\left(h^\theta + \frac{q^n}{1-q} + \frac{r_1^n}{1-r_1}\right)\|f\|_{W^{1,1}(0,T;H_{\frac{1}{2}})} \quad (n \geqslant 1,\ h < h_1). \tag{7.24}$$

Proof. From (7.22) we deduce that, for each $h < h_1$, the operator Λ_h defined by (3.20) may be restricted to an operator from $\mathcal{L}(W_h^1 \times W_h^1)$ which is uniformly contractive in h. Indeed, since the semigroup $\mathbb{S}_h$ verifies (7.6) in $W_h^1 \times W_h^1$, it follows that, for any $n \geqslant 1$,

$$\|\Lambda_h^n U_h^0 - \Lambda_h^n U_h^1\|_X \leqslant M_1 r_1^n \|U_h^0 - U_h^1\|_X \qquad (U_0^h, U_1^h \in W_h^1 \times W_h^1, \qquad h \in (0, h_1)). \tag{7.25}$$

Now, the rest of the proof follows as in Theorem 6.4. □

Remark 7.6. *When applying our method to simulation purposes we do not have access to the exact solution* $(u_h, \dot{u}_h)$*. More precisely, we need to discretize equations* (1.4) *and* (1.6) *with respect to the time variable. Using error estimates for full space-time discretizations corresponding to those in Proposition 5.1 it would be possible, after a careful numerical analysis, to obtain convergence rates for the periodic solutions based on full discretizations. However, since the lack of uniform decay can be a consequence of the time discretization, the numerical viscosity introduced for the full space-time discretization of* (1.4) *should depend on the time discretization step too (see [13] for a space-time discretization scheme which conserve the uniform decay of the energy for a class of dissipative equations).*

Remark 7.7. *Solving numerically the linear Helmholtz equation* (7.3)-(7.4) *at high wavenumbers k remains as one of the most difficult tasks in scientific computing. This is due to the highly oscillatory character of the solutions and to the non-hermitian and non-positive structure of the discrete system's coefficient matrix. If the mesh size of a finite element discretization is h, to ensure an accurate numerical solution it is generally needed to impose restrictive condition like* $k^2 h < 1$ *(see [1]). This represents the so-called "pollution effect" (see, for instance, [2]) and leads to a huge system of linear equations. The resulting system being highly indefinite for large wavenumbers, many iterative techniques such as the conjugate gradient and multigrid methods are not capable of solving it.*

The problem of high wave numbers in the Helmholtz equation (7.3) *would correspond, in the case of periodic solutions problem, to small periods* T*. As it can be deduced from our error estimates, all the constants depend on the period* T*. In fact a degeneracy of the convergence process as* T *is becoming small has been detected in our computations (see Table 1 in Section 8.1.1). The aim of this section was to illustrate the fact that a source term with only one time frequency behaves better than a general one from the point of view of the fixed point method convergence. Although some insights on the resolution of the Helmholtz equation can be obtained from our results, we would like to emphasize that our study is concerned with a fixed wavenumber. To study the dependence of the convergence process on the wavenumber, or equivalently, to see what is happening as* T *tends to zero, would require additional estimates and it remains to be done.*

Remark 7.8. *An interesting open question concerns the existence of a periodic function* f *for which the approximation process fails to converge without numerical viscosity. As in Theorem 3.5, one can easily deduce the existence of a discrete periodic solution* $\widehat{U}_h \in C([0,T]; V_h^2)$ *of* (3.8) *under the weaker hypothesis* $f_h \in L^1(0,T;V_h)$*. The failure of* *convergence of the family* $(\widehat{U}_h)_{h>0}$ *to the finite energy periodic solution of the continuous equation is equivalent to the unboundedness of the corresponding initial data* $(\widehat{U}_h^0)_{h>0}$ *in* $X_{\frac{1}{2}}$. *To analyze this possibility, let us consider that* f_h *has the following Fourier expansion*

$$f_h(t) = \sum_{1\leqslant n\leqslant N(h)} a_{hn} e^{i\frac{2\pi n}{T}t} \varphi_{hn}, \tag{7.26}$$

where $(a_{hn})_n$ *is a uniformly bounded sequence in* ℓ^1. *As in the proof of Theorem 7.2 (see* (7.14)*), it follows that the initial data corresponding to the periodic solution* $\widehat{U}_h$ *is given by*

$$\widehat{U}_h^0 = \sum_{1\leqslant n\leqslant N(h)} a_{hn} \left(i\frac{2\pi n}{T} - \mathbb{A}_h \right)^{-1} \begin{bmatrix} 0 \\ \varphi_{hn} \end{bmatrix}. \tag{7.27}$$

Note that a necessary condition for the unboundedness of the family $(\widehat{U}_h^0)_{h>0}$ *is the unboundedness of the norm of the resolvent* $\left(i\frac{2\pi n}{T} - \mathbb{A}_h\right)^{-1}$ *as* n *goes to infinity and* h *tends to zero. By taking into account the decay* (3.16) *of the discrete energy and the following resolvent estimates (see [26][Corollary 2.3. and Remark 2.2.8])*

$$\frac{M}{\omega(h)} \geqslant \left\| \left(i\frac{2\pi n}{T} - \mathbb{A}_h \right)^{-1} \right\| \geqslant \frac{1}{\min_{\lambda_h\in\sigma(\mathbb{A}_h)} \left| i\frac{2\pi n}{T} - \lambda_h \right|}, \tag{7.28}$$

we deduce that the sequence of initial data $(\widehat{U}_h^0)_{h>0}$ *defined by* (7.27) *is unbounded only if* $\lim_{h\to 0}\omega(h)$. *Note that the not uniform decay rate of the discrete semigroup is equivalent to the fact that the distance between the spectrum* $\sigma(\mathbb{A}_h)$ *of the operator* $\mathbb{A}_h$ *and the imaginary axis tends to zero as* h *goes to zero.*

On the other hand, the second inequality from (7.28) *implies that a sufficient condition for the failure of the convergence process is a kind of "asymptotic resonance phenomenon" expressed by the fact that the distance between the spectrum of* $\mathbb{A}_h$ *and the set* $\mathcal{E} = \left\{ i\frac{2\pi n}{T} \mid n \geqslant 1 \right\}$ *tends to zero as* h *goes to zero. To the best of our knowledge, such precise information on the discrete spectrum* $\sigma(\mathbb{A}_h)$ *is not available in the literature and it is not easy to obtain. The resonance phenomenon depends, for instance, on the values of the period* T *and on the velocity at which the high eigenvalues* λ_h *approach the imaginary axis. This could explain why it is difficult to prove theoretically that there are cases in which the convergence fails and almost impossible to detect them numerically. We refer the interested reader to [11] for an analysis of a similar problem in the context of a weakly dissipated hybrid system.*

8 Numerical experiments

In this section we numerically illustrate the theoretical results obtained in the previous sections by considering internally damped wave equations in one or two dimensional domains and with different periodic force terms. A finite elements discretization of the considered wave equations leads to a system of the form (4.3)-(4.4). For each periodic forcing term regular enough, Theorem 4.3 gives the existence of an initial data $\widehat{U}_h^0$ for which the corresponding solution of (4.3) is periodic with the same period as the forcing term. To approximate this initial data we iterate the operator $\Lambda_{h\vartheta}$ introduced by (4.5) for different values of η and ϑ. Therefore, for a given $h > 0$ and an initial data $U^0 \in X_1$, we compute $\Lambda_{h\vartheta}^n \Pi_h U^0$ until the difference between two consecutive iterations

$\|\Lambda^n_{h\vartheta}\Pi_h U^0 - \Lambda^{n-1}_{h\vartheta}\Pi_h U^0\|_X$ becomes smaller than a desired precision ϵ or until the number of iterations n becomes larger than a prescribed number N. In the case when the initial data $\widehat{U}^0_h$ corresponding to the periodic source term is known, we also compute the error between the current iteration and the known initial data $\|\Lambda^n_{h\vartheta}U^0_h - \Pi_h\widehat{U}^0\|_X$.

In order to implement the algorithm described above we need to discretize (4.1) in time too. For this purpose, we use a classical Newmark scheme of parameters $\gamma = 0.5$ et $\beta = 0.25$ which is known to be a second order (in time) unconditionally stable method (see, for instance, [16]).

8.1 One dimensional wave equation

Consider the following one-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \dfrac{\partial^2 w}{\partial x^2}(t,x) + a(x)\dot{w}(t,x) = f(t,x), & ((t,x) \in (0,\infty)\times(0,1)), \\ w(t,0) = w(t,1) = 0, & (t \in (0,\infty)) \end{cases} \tag{8.1}$$

where $a : [0,1] \to \mathbb{R}$ is a nonnegative regular function which is strictly positive in a subdomain $\omega \subset (0,1)$ and $f \in \mathcal{C}([0,\infty); L^2(0,1))$ is a periodic function of period T such that $f_{|(0,T)} \in W^{1,1}(0,T; H^1_0(0,1))$.

Denote $H = L^2(0,1)$ and let $A : \mathcal{D}(A) \to H$ be the operator defined by

$$\mathcal{D}(A) = H^2(0,1) \cap H^1_0(0,1), \qquad A\varphi = -\frac{d^2\varphi}{dx^2} \qquad (\varphi \in \mathcal{D}(A)).$$

Let $U = H = L^2(0,1)$ and let $B \in \mathcal{L}(U,H)$ be an input operator given by

$$Bu = \sqrt{a}u \qquad (u \in U).$$

Using these notations, equation (8.1) can be written in the abstract form (2.13). Moreover, the domain of the operator $A^{\frac{1}{2}}$ is given by $H_{\frac{1}{2}} = H^1_0(0,1)$.

To construct a family of finite-dimensional subspaces of $H_{\frac{1}{2}}$ we consider a uniform discretization $\mathcal{I}_h$ of the interval $(0,1)$ formed by N equidistant points and we denote $h = 1/(N+1)$. For each h we define V_h by

$$V_h = \left\{\varphi \in \mathcal{C}(0,1) \,|\, \varphi_{|I} \in P_2(I) \text{ for every } I \in \mathcal{I}_h, \quad \varphi(0) = \varphi(1) = 0\right\}. \tag{8.2}$$

In (8.2) we denote by $P_2(I)$ the set of polynomial functions of degree 2 on I. Moreover, it is well known that the estimates (3.1)-(3.2) are satisfied by the orthogonal projector $\pi_h : H^1_0(0,1) \to V_h$ for $\theta = 2$ (see, for instance, [20, p. 96-97]).

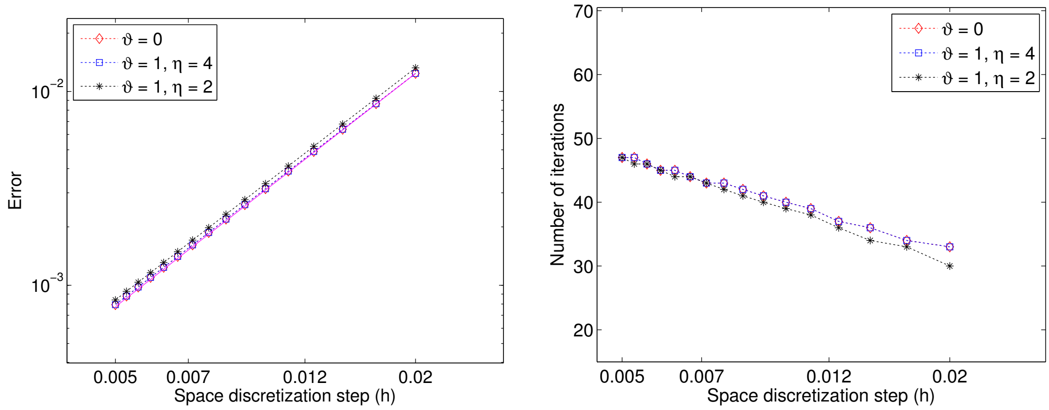
8.1.1 A monochromatic-type periodic function

A first numerical test consists in approximating the periodic solution of (8.1) when the periodic source term is a monochromatic-type function f given by

$$f(t,x) = (-k^2 + \pi^2)\sin(\pi x)\cos(kt) - ka(x)\sin(\pi x)\sin(kt). \tag{8.3}$$

Taking $k = \frac{2\pi}{T}$, it is easy to verify that $w(t,x) = \sin(\pi x)\cos(kt)$ is the corresponding T-periodic solution of (8.1). Therefore, the fixed point of the operator Λ introduced in Theorem 2.3 is given by $\widehat{U}^0 = \begin{bmatrix}\sin(\pi x)\\ 0\end{bmatrix}$.

In Figure 2 we display the results obtained for the function f given by (8.3) with $T = \frac{\pi}{2}$ and $\omega = (0.2, 0.8)$. More precisely, Figure 2(a) shows the error $\|\widehat{U}^0 - \Lambda_{h\vartheta}^n U_0\|_X$ between the fixed point $\widehat{U}^0$ of the operator Λ and the $n(h)$-th iteration in the fixed point algorithm. The values of $n(h)$ are displayed in Figure 2 and are chosen such that the norm of the difference between two consecutive iterations in the fixed point algorithm is smaller than $\epsilon = h^3$. These numerical results illustrate well the estimates proven in Section 7, in Figure 2(a) clearly distinguishing the convergence rate $C_0 h^2$. More exactly, the solid line is nothing else than the curve $e(h) = C_0 h^2$ where C_0 is a constant chosen such that that the values of the error $\|\widehat{U}^0 - \Lambda_{h\vartheta}^n U_0\|_X$ for $h = 0.02$, $\vartheta = 0$ and $e(0.02)$ coincide. Remark that these errors are less influenced by the presence of the viscosity. As expected, the number of iterations needed to achieve the precision h^3 increases when h decreases with no important differences for different values of ϑ and η.



(a) (b)

Figure 2: (a) Error for a fixed period $T = \frac{\pi}{2}$ and for values of h distributed between $1/200$ and $1/50$. (b) The number of iterations necessary to achieve a precision $\epsilon = h^3$ in the fixed point algorithm for $T = \frac{\pi}{2}$ and values of h distributed between $1/200$ and $1/50$.

Table 1 displays the number of iterations $n(h)$ and the error $\|\widehat{U}_0 - \Lambda_{h\vartheta}^{n(h)} U_0\|_X$ for values of the period T listed in its first row. The periodic source term is given by (8.3), $\omega = (0.2, 0.8)$, the space discretization step is $h = 0.005$, the time discretization step is $\Delta t = 1.5h$ and, as in the previous paragraph, $n(h)$ is the minimal number of iterations such that the norm of the difference between two consecutive iterations in the fixed point algorithm is smaller than $\epsilon = h^3$.

Period T	0.10	0.15	0.30	0.45	0.60	0.80	1.00
$n(h)$ for $\vartheta = 0$	10000	4203	1518	794	491	238	95
$n(h)$ for $\vartheta = 1$, $\eta = 4$	10000	4183	1510	791	490	237	94
$n(h)$ for $\vartheta = 1$, $\eta = 2$	660	448	233	155	108	90	71
Error	0.0873	0.0370	0.0096	0.0043	0.0026	0.0015	0.0011

Table 1: Number of iterations $n(h)$ and error $\|\widehat{U}_0 - \Lambda_{h\vartheta}^{n(h)} U_0\|_X$ for different values of T.

We remark in table 1 that the number of iterations $n(h)$ increases substantially when the period T becomes small which illustrates the difficulties arising from the Helmholtz equation with high wavenumbers (see Remark 7.7). While there is no significant difference between cases $\vartheta = 0$ and $\vartheta = 1$, $\eta = 4$, a much better behavior is obtained when $\vartheta = 1$, $\eta = 2$. This suggests that the presence of a well chosen numerical viscosity ameliorates the convergence of the discrete solutions of Helmholtz's equation with high wavenumbers. However, a theoretical study of the error in such viscous approximations remains to be done.

8.1.2 A general periodic function

We consider a periodic source f as follows

$$\begin{aligned} f(t,x) =& \alpha t(T-t)\left(6(T-t)^2 - 18t(T-t) + 6t^2\right) x^3(1-x)^3 \\ & - \alpha\left(1 + t^3(T-t)^3 x(1-x)\right)\left(6(1-x)^2 - 18x(1-x) + 6x^2\right) \\ & + \alpha 3t^2(T-t)^2(T-2t)a(x)x^3(1-x)^3, \end{aligned} \tag{8.4}$$

for $x \in (0,1)$ and $t \in (0,T)$ and being extended by periodicity to $(0,\infty)$. The parameter α is chosen such that $\max\{f(t,x) | (t,x) \in (0,T)\times(0,1)\} = 1$. The corresponding periodic solution of (8.1) is $w(t,x) = \alpha\left(1 + t^3(T-t)^3\right) x^3(1-x)^3$ and the fixed point of the operator Λ is $\widehat{U}^0 = \begin{bmatrix} \alpha x^3(1-x)^3 \\ 0 \end{bmatrix}$.

In Figure 3 we display the results obtained for the function f given by (8.4) with $T = 1.5$ and $\omega = (0.2, 0.8)$. More precisely, in Figure 3(a) is displayed the error $\|\widehat{U}^0 - \Lambda^n_{h\vartheta} U_0\|_X$ between the fixed point $\widehat{U}^0$ of the operator Λ and the $n(h)$-th iteration in the fixed point algorithm. The values of $n(h)$ are shown in Figure 3(b) and are chosen such that the norm of difference between two consecutive iterations in the fixed point algorithm is smaller than $\epsilon = h^3$. The results in Figure 3 are similar to the ones obtained for a monochromatic-type function in Section 8.1.1. As expected, the number of iterations necessary to achieve precision h^3 increases when h goes to zero. Remark that the presence of the viscosity term don't change too much the performances of our algorithm. This is in concordance with estimations given by Theorems 6.4 and 6.8 which proves the convergence of the sequence of iterations in the fixed point algorithm to the fixed point of the operator Λ. The principal difference is that when the viscosity is present and $\eta = \theta$ the number of iterations needed to achieve a precision $\epsilon = h^3$ in the fixed point algorithm is slightly smaller. As in Figure 2, the solid line in Figure 3 represents the curve $e(h) = C_0 h^2$ with a calibrated value of C_0.

8.2 Two dimensional wave equation

Let $\Omega \subset \mathbb{R}^2$ be an open and non-empty set with regular boundary or a square. We consider the following two-dimensional wave equation

$$\begin{cases} \ddot{w}(t,x) - \Delta w(t,x) + a(x)\dot{w}(t,x) = f(t,x), & ((t,x) \in (0,\infty)) \times \Omega \\ w(t,x) = 0, & ((t,x) \in (0,\infty) \times \partial\Omega) \end{cases} \tag{8.5}$$

where $a \in \mathcal{C}^1(\Omega)$ is a nonnegative function which is strictly positive in an open and non-empty subdomain $\omega \subset \Omega$. Moreover, the function $f \in \mathcal{C}([0,\infty); L^2(\Omega))$ on the right hand side of (8.5) is periodic of period T and verifies $f_{|(0,T)} \in W^{1,1}(0,T; H^1_0(\Omega))$.

As for the one-dimensional case, we denote $H = L^2(\Omega)$ and we define the operator $A : \mathcal{D}(A) \to H$ by

$$\mathcal{D}(A) = H^2(\Omega) \times H^1_0(\Omega), \qquad A\varphi = -\Delta\varphi, \qquad (\varphi \in \mathcal{D}(A)).$$

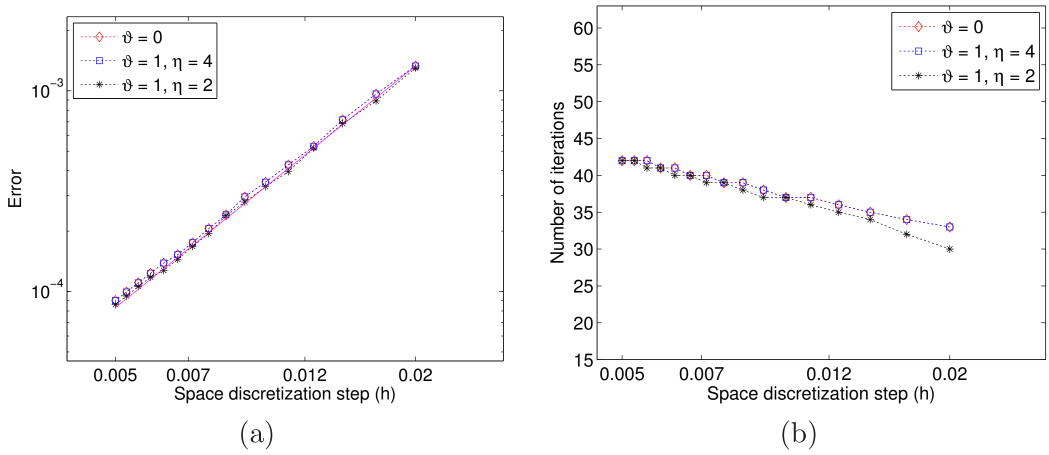


Figure 3: (a) Error for a fixed period $T = 1.5$ and for values of h distributed between 1/200 and 1/50. (b) The number of iterations necessary to achieve a precision $\epsilon = h^3$ in the fixed point algorithm for $T = 1.5$ and values of h distributed between 1/200 and 1/30.

Similarly, denote $U = H$ and let $B \in \mathcal{L}(U, H)$ be an input operator defined by $Bu = \sqrt{a}u$. With these notations (8.5) can be written as (2.13). As in Section 8.1 we denote $H_{\frac{1}{2}} = \mathcal{D}(A^{\frac{1}{2}})$. To construct a family of finite-dimensional subspaces of $H_{\frac{1}{2}}$ we consider triangulations $\mathcal{T}_h$ of the set Ω, with h being the maximal size of the triangles. Then, for each $h > 0$ we denote

$$V_h = \left\{\varphi \in \mathcal{C}(\Omega) \mid \varphi_{|T} \in P_1(T) \text{ for every } T \in \mathcal{T}_h, \quad \varphi = 0 \text{ on } \partial\Omega\right\}.$$

We easily recognize in V_h the space of P_1 finite elements. It is well-known (see, for instance, [22]) that the projector π_h on this finite elements space verifies (3.1)-(3.2) with $\theta = 1$.

In the remaining part of this sub-section we discuss two numerical tests, in the first one Ω being a square and the periodic solution being known explicitly, and in the second case the domain Ω is a convex arbitrary regular domain, the periodic solution being unknown.

8.2.1 Wave equation in a square

Let $\Omega = (0, 1) \times (0, 1)$ be the unit square. For this domain, following the ideas in one-dimensional case, we can build the exact periodic solution corresponding to a well-chosen periodic function f.

Firstly, we consider a uniform triangulation $\mathcal{T}_h$ of Ω considering $N = 40$ discretization points on each side of the square, which produces 3042 triangles. Also, we consider a nonnegative function $a \in \mathcal{C}^1(\Omega)$ which is strictly positive along two consecutive sides of the square Ω (see Figure 4). On this mesh we solve system (4.1) for $\vartheta \in \{0, 1\}$ and $\eta \in \{1, 2\}$. The time step is set to $\Delta t = 1/50$.

The numerical test considered in this case consists in approaching the periodic solution

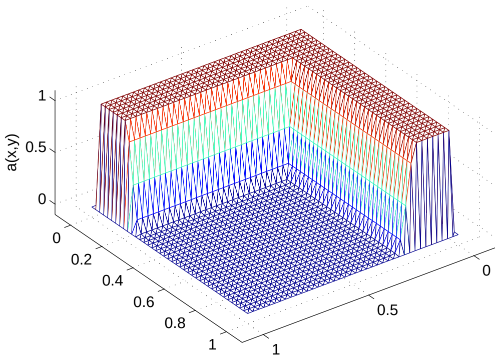


Figure 4: The function $a \in \mathcal{C}^1(\Omega)$.

(i.e. the corresponding initial data) associated to the periodic function

$$\begin{aligned} f(t,x,y) =&\alpha\big(6t(T-t)^3 - 18t^2(T-t)^2 + 6t^3(T-t)\big)x^3(1-x)^3y^3(1-y)^3 \\ &- \alpha\left(1+t^3(T-t)^3\right)\left(6x(1-x)^3 - 18x^2(1-x)^2 + 6x^3(1-x)\right)y^3(1-y)^3 \\ &- \alpha\left(1+t^3(T-t)^3\right)\left(6y(1-y)^3 - 18y^2(1-y)^2 + 6y^3(1-y)\right)x^3(1-x)^3 \\ &+ \alpha\big(3t^2(T-t)^3 - 3t^3(T-t)^2\big)a(x,y)x^3(1-x)^3y^3(1-y)^3, \end{aligned}$$

where $\alpha > 0$ is such that $\max\{f(t,x,y) \mid (t,x,y) \in (0,T)\times(0,1)\times(0,1)\} = 1$ and $T = 0.6$. It is easy to see that the periodic solution associated to this function is

$$w(t,x,y) = \alpha\left(1+t^3(T-t)^3\right)x^3(1-x)^3y^3(1-y)^3$$

and hence the initial data which gives the periodic solution (i.e. the fixed point of operator Λ) to approximate is $\widehat{U}^0 = \begin{bmatrix} \alpha x^3(1-x)^3y^3(1-y)^3 \\ 0 \end{bmatrix}$.

In Figure 5 we display the error (in the energy norm) between this known initial data, corresponding to the periodic solution, and the current iteration $\Lambda^n_{h\vartheta}U^0_h$ in the fixed point algorithm.

Remark that the presence of the viscosity term accelerates the convergence, the number of iterations necessary to obtain the desired precision in the fixed point algorithm being twice smaller in the case when $\vartheta = 1$ and $\eta = 1$ than in the case when $\vartheta = 0$.

8.2.2 Wave equation in a convex domain with $\mathcal{C}^1$ boundary

Let $\Omega \subset \mathbb{R}^2$ be a convex domain with $\mathcal{C}^1$ boundary and $\omega \subset \Omega$ be an open and non-empty subset as described in Figure 6 (a). We consider a uniform triangulation $\mathcal{T}_h$ of the domain Ω formed by 928 nodes and 1725 triangles (see Figure 6(b)).

We consider the following periodic function $f \in \mathcal{C}([0,\infty); L^2(\Omega))$

$$f(t,x) = \psi(x)\cos\left(\frac{2\pi t}{T}\right), \tag{8.6}$$

where T is the period and ψ is the solution of the following elliptic problem

$$\begin{cases} \Delta\psi(x) = 1, & (x \in \Omega) \\ \psi(x) = 0, & (x \in \partial\Omega). \end{cases}$$

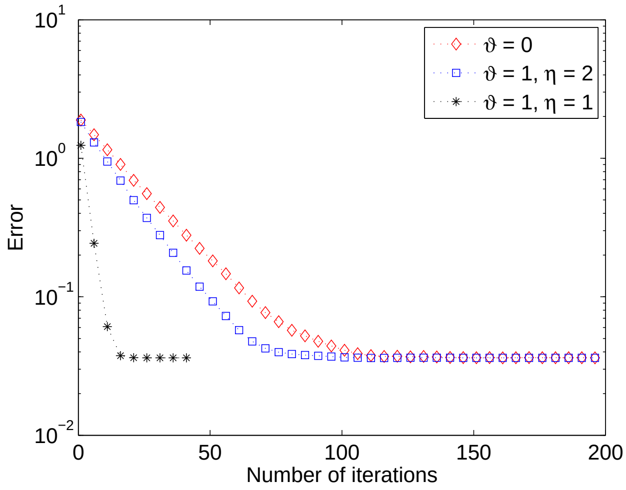
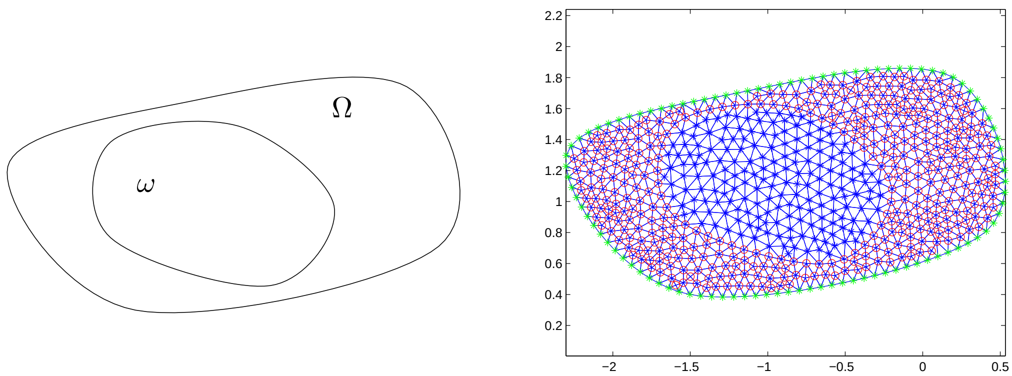


Figure 5: (a) Evolution of the error in the fixed point algorithm as a function of the iteration's number.



(a) (b)

Figure 6: (a) Domains Ω and ω. (b) Triangulation of the domain Ω: by circles we design the points in $\Omega \setminus \omega$, and by stars the points in ω.

Since in this case we do not know the explicit periodic solution associated to the periodic function f we cannot evaluate the error between the initial data corresponding to the periodic solution and the iterations in the fixed point algorithm used to approach this initial data. Hence, in Figure 7 we display the difference between two consecutive iterations in the fixed point algorithm for $\vartheta \in \{0,1\}$ and $\eta = 1, 2$. Notice that in all the numerical experiments of this subsection we considered $T = 0.6$ and $\Delta t = 0.05$.

The fixed points approached after the number of iterations shown in Figure 7 (the stopping criterium used in the fixed point was the difference between two consecutive iterations is smaller than $\epsilon = 10^{-5}$ and the maximal number of iterations was set to 200) are displayed in Figure 8. Remark that the fixed points of operator $\Lambda_{h\vartheta}$ displayed in Figure 8 are very similar in the case without viscosity (a) and in the cases with viscosity (b, c). However, we can observe that the second component of the initial data (the initial velocity) has less oscillations when $\vartheta = 1$ and $\eta = 1$ (c).

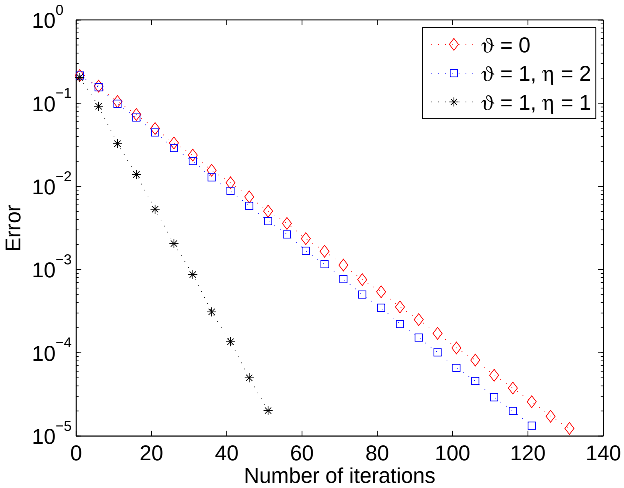


Figure 7: The difference in the energy norm between two consecutive iterations in the fixed point algorithm.

In conclusion, all our numerical examples illustrate the convergence of the family $(\Lambda^n_{h\vartheta}\Pi_h U^0)_{n,h}$ to the fixed point of the operator Λ when n goes to infinity and h goes to zero which is in consonance with theoretical results proven in Theorem 6.4 from Section 6. We recall that the convergence of the iterative method was shown under the regularity hypothesis $f_{|(0,T)} \in W^{1,1}(0,T;H_{\frac{1}{2}})$. What happens for less regular periodic source terms (for which no error estimates as in Proposition 5.1 can be provided) will make the subject of a future research work.

On the other hand, when a viscosity is added, it has been observed that a diminished number of iterations is needed to reach the desired precision. The reduction is noticeable mostly in the 2D simulations. This is in concordance with the error estimates obtained in Theorem 6.8 which are slightly better than the ones in Theorem 6.4.

Finally, since the differences between two consecutive iterations become very small in all our examples, it seems that we do have convergence of the fixed points of the discrete operator $\Lambda_{h\vartheta}$ to the fixed point of Λ even in the case $\vartheta = 0$. Theoretically, this property was proved only for the monochromatic source terms or in the presence of viscosity (Theorem 7.2 and Corollary 6.9 respectively) and the general case remains an open problem.

Acknowledgments: The authors are grateful to Enrique Zuazua for several interesting suggestions and comments.The second author was partially supported by Grant PN-II-ID-PCE-2011-3-0257 of the Romanian National Authority for Scientific Research, CNCS-UEFISCDI and by Grant MTM2011-29306 funded by MICINN (Spain). Part of this work was done during the second author's visit to Basque Center for Applied Mathematics and he would like to thank the members of the center for their kindness and warm hospitality. This work was done while the third author was a postdoctoral visitor at Basque Center for Applied Mathematics, supported by a grant from the Brazilian Government, CAPES, proc. n^o 4195/10-8. He would like to express his sincere thanks to Professor Enrique Zuazua for the invitation to spend one year at BCAM and to all people of the center for their friendship and hospitality.

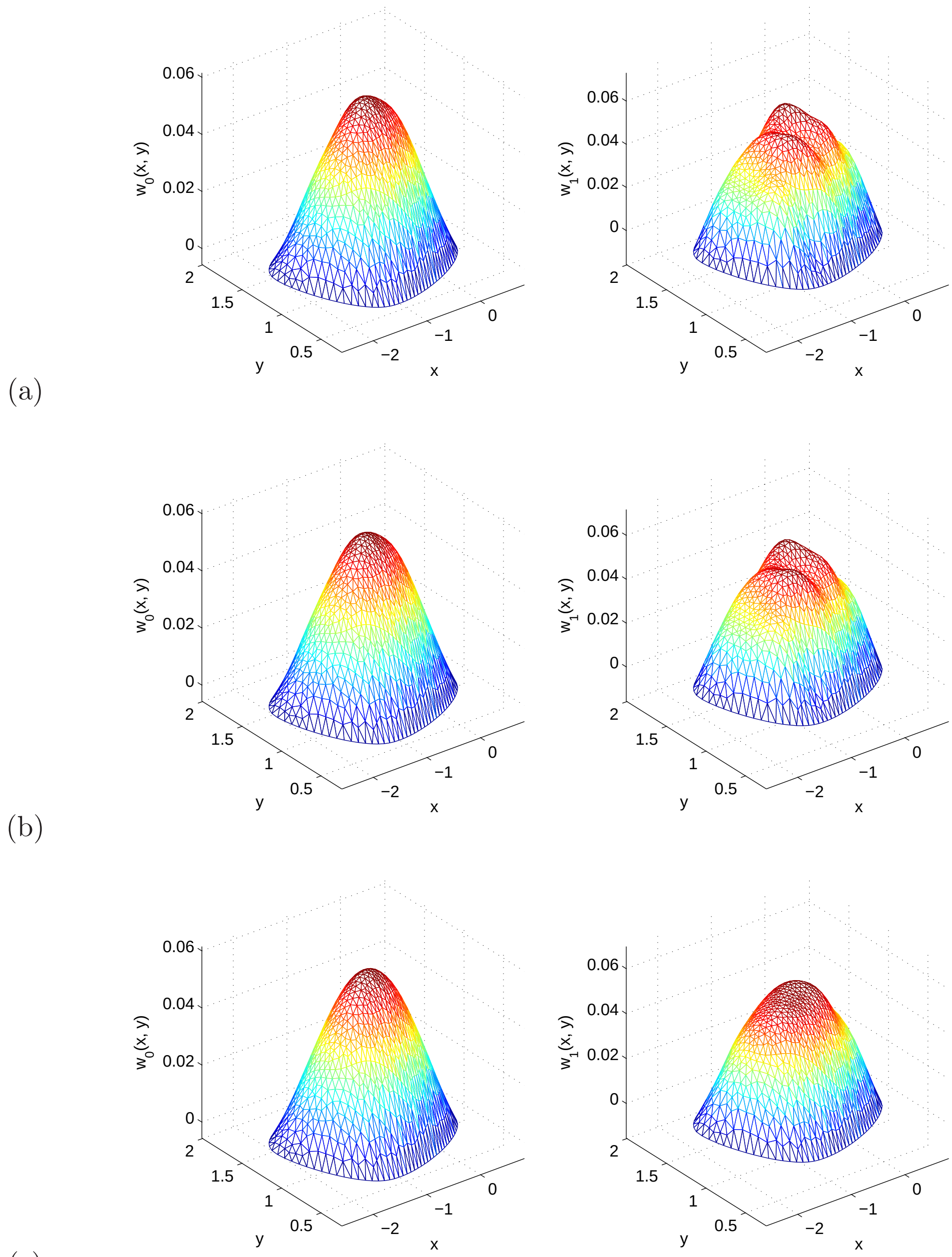


Figure 8: Approached fixed point. (a) $\vartheta = 0$. (b) $\vartheta = 1$, $\eta = 2$. (c) $\vartheta = 1$, $\eta = 1$.

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